Letzte Mathematik 1 Vorlesung 31. 1. 2024

Integralrechnung

$\int \sin(x) \cdot \cos(x)\, dx$ $\qquad f(x) = \sin(x)$ $\quad g'(x) = \cos(x)$

$\qquad f'(x) = \cos(x)$ $\quad g(x) = \sin x$

$$\int \underset{f(x)}{\sin(x)} \cdot \underset{g'(x)}{\cos(x)}\, dx = \sin(x) \cdot \sin(x) - \int \cos(x) \cdot \sin(x)\, dx \quad \Big| + \int \cos(x) \cdot \sin(x)\, dx$$

$$2 \int \sin(x) \cdot \cos(x)\, dx = \sin^2(x)$$

$$\Rightarrow \int \sin(x) \cos(x)\, dx = \frac{1}{2} \sin^2(x) + C, \quad C \in \mathbb{R}$$

$$\int \ln(x)\, dx = \int 1 \cdot \ln(x)\, dx$$

Hinweis

$f(x) = \ln(x)$

$f'(x) = \frac{1}{x}$

$g'(x) = 1$

$g(x) = x$

$$= \ldots$$

$$= x(\ln(x) - 1) + C, \quad C \in \mathbb{R}$$

2) Integration durch Substitution

Differentialrechnung: $y = f(g(x)) \Rightarrow y' = f'(g(x)) \cdot g'(x)$

Bp: $\int \sqrt{x^2 + 2} \cdot 2x\, dx$

Substitution $z = x^2 + 2$ $\qquad z' = \frac{dz}{dx} = 2x \Rightarrow dz = 2x\,dx$

$$\int \sqrt{z}\, dz = \int z^{\frac{1}{2}}\, dz$$

$$= \frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{z^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3} \cdot z^{\frac{3}{2}} = \frac{2}{3}\sqrt{z^3}$$

Mit $z = x^2 + 2$ $\qquad \int \sqrt{x^2+2} \cdot 2x\, dx = \frac{2}{3}\sqrt{(x^2+2)^3} + C \quad C \in \mathbb{R}$

Substitutionsregel

1) $z = g(x)$ $\quad dz = g'(x)\cdot dx$ $\quad dx = \frac{dz}{g'(x)}$

2) $\int f(x)\cdot g'(x)\,dx = \int f(z)\,dz$

Bp: 1) $\int x \cdot e^{-x^2}\,dx$ $\qquad z = -x^2$ $\qquad z' = \frac{dz}{dx} = -2x$

$dz = -2x\,dx$

$dx = \frac{dz}{-2x}$

$$= \int e^{\overset{z}{-x^2}} \cdot \underbrace{x\,dx}_{-\frac{1}{2}\cdot \underbrace{dz}_{-2x\cdot dx}} \qquad \frac{dz}{-2x}$$

$$= -\frac{1}{2}\int e^z\,dz = -\frac{1}{2}\cdot e^z \underset{\text{Rücksubst.}}{=} -\frac{1}{2}e^{-x^2} + C,\ C\in\mathbb{R}$$

2) $\int \frac{1}{2x+3}\,dx$ $\qquad z = 2x+3$ $\qquad z' = \frac{dz}{dx} = 2$ $\qquad dz = 2dx$

$$\frac{1}{2}\int \frac{1}{z}\,dz$$

$$= \frac{1}{2}\ln|z| \underset{\text{Rücksubst.}}{=} \frac{1}{2}\ln|2x+3| + C,\ C\in\mathbb{R}$$

3) $\int \tan(x)\,dx$ $\qquad$ Es gilt $\tan(x) = \frac{\sin(x)}{\cos(x)}$

$$= \int \frac{\sin(x)\,dx}{\cos(x)} = -1\int \frac{1}{z}\,dz$$

$z = \cos(x)$ $\qquad z' = \frac{dz}{dx} = -\sin(x)$

$dz = -\sin(x)\,dx$

$$= -\ln|z| \underset{\text{Rücksubst.}}{=} -\ln|\cos(x)| + C,\ C\in\mathbb{R} \quad x \neq \frac{\pi}{2} + k\pi \quad k\in\mathbb{Z}$$

Bem: In Formelsammlungen findet man umfangreiche Integraltabellen

Hinweis auf weitere Integrationsmethoden

Integration von echt gebrochen rationalen Funktionen

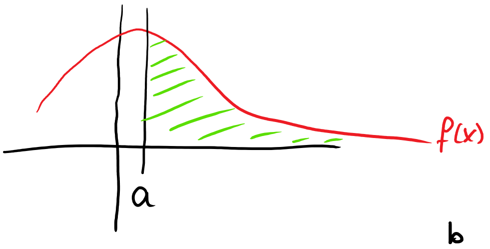
Partialbruchzerlegung

Ist ein Integral nicht geschlossen lösbar (also in endlich vielen Schritten), dann gibt es numerische Verfahren

- Rechtecksapproximation
- Trapezregel
- Simpson-Regel

Uneigentliche Integrale

$$\int_a^{\infty} f(x)\,dx \qquad \int_{-\infty}^{b} f(x)\,dx \qquad \int_{-\infty}^{+\infty} f(x)\,dx$$

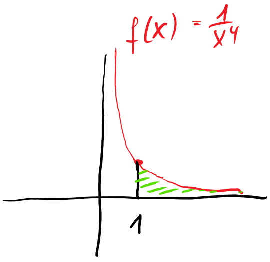


$$\int_a^{\infty} f(x)\,dx = \lim_{b_n \to \infty} \int_a^{b_n} f(x)\,dx$$

ebenso: $\lim\limits_{a_n \to -\infty} \int_{a_n}^{b} f(x)\,dx$

$$\lim_{\substack{a_n \to +\infty \\ b_n \to -\infty}} \int_{a_n}^{b_n} f(x)\,dx$$

Bp: 1) $\int_1^{\infty} \frac{1}{x^4}\,dx$



$$\int_1^{b} \frac{1}{x^4}\,dx = \left[-\frac{1}{3}\cdot x^{-3}\right]_1^{b}$$

$$= \frac{b^{-3}}{-3} - \frac{1^{-3}}{-3}$$

$$= -\frac{1}{3b^3} + \frac{1}{3}$$

$$\lim_{b\to\infty}\left(-\frac{1}{3b^3} + \frac{1}{3}\right)$$

$$= \lim_{b\to\infty} -\frac{1}{3b^3} + \lim_{b\to\infty} \frac{1}{3}$$

$$= 0 + \frac{1}{3} = \frac{1}{3}$$

2) $\int_0^{\infty} e^{-x}\,dx$

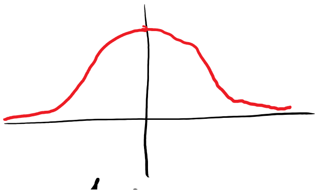
$$\downarrow$$

$$\lim_{b\to\infty}\left(\int_0^{b} e^{-x}\,dx\right)$$

$$= \lim_{b\to\infty}\left[-e^{-x}\right]_0^{b} = \lim_{b\to\infty}\left[-e^{-b} + e^{0}\right]$$

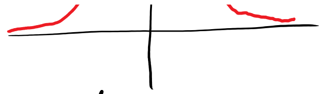
$$= \lim_{b\to\infty}\left(\underbrace{-\frac{1}{e^b}}_{\to 0} + 1\right) = 1 \qquad \int_0^{\infty} e^{-x}\,dx = 1$$

Hinweis auf SS:



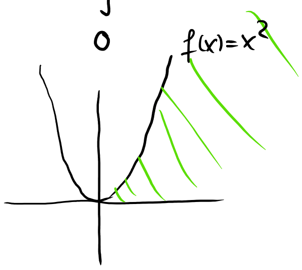
$\int_{-\infty}^{+\infty}$ Gaußkurve $= 1$

0



$-\infty$

3) $\int_0^\infty x^2\,dx$



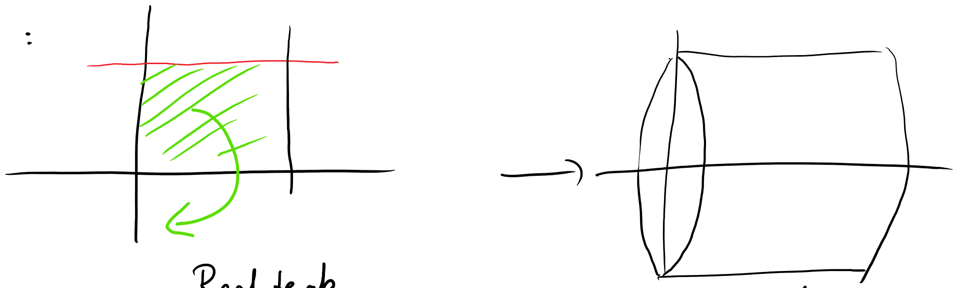
Stammfunktion: $\frac{1}{3}x^3$

$$\lim_{b\to\infty}\int_0^b x^2\,dx = \lim_{b\to\infty}\left[\frac{1}{3}b^3 - 0\right] = \lim_{b\to\infty}\frac{1}{3}b^3 = \infty$$

Das uneigentliche Integral konvergiert nicht, es ist divergent

Anwendungsgebiete der Integralrechnung

Vorschau:



Rotationsvolumen

1) Flächeninhalt

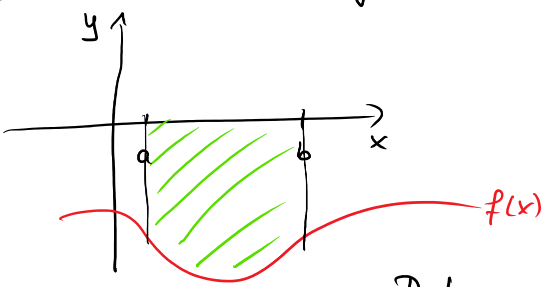
verschobene Normalparabel

Bp: $f(x) = x^2 - 2x + 3$

$$A = \int_0^3 (x^2 - 2x + 3)\,dx \text{, da } f(x) \geq 0 \text{ auf } [0,3]$$

$$= \left[\frac{1}{3}x^3 - x^2 + 3x\right]_0^3 = 9 - 9 + 9 - 0 = 9 \text{ FE}$$

Folgender Fall: $f(x) \leq 0$ auf $[a,b]$

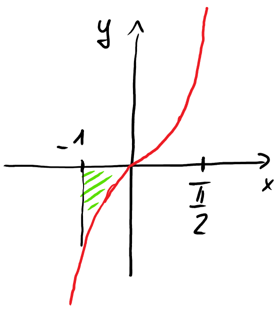


$\int_a^b f(x)\,dx$ liefert hier einen negativen Wert

Daher: $A = \left| \int_a^b f(x)\,dx \right|$

Bp: $\int_{-1}^{0} \tan(x)\,dx$

$= \int_{-1}^{0} \frac{\sin(x)}{\cos(x)}\,dx$



$$A = \left| \int_{-1}^{0} \tan(x)\,dx \right|$$

$$= \left| \left[-\ln|\cos(x)| \right]_{-1}^{0} \right|$$

$$= \left| -\ln|\cos(0)| + \ln|\cos(-1)| \right|$$

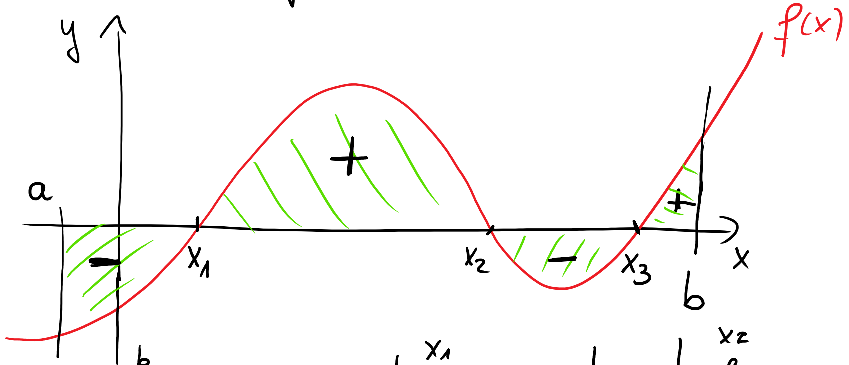
$\cos 0 = 1$
$\ln 1 = 0$

$$= \left| -0 + \ln(0.54) \right| = \left| -0.616 \right|$$

$$= 0.616 \text{ FE}$$

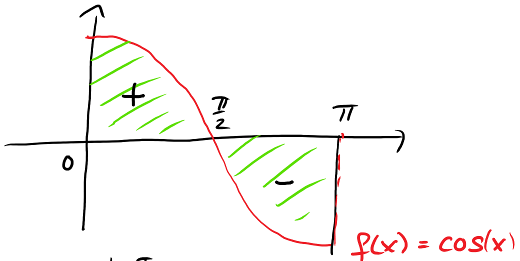
Falls der Kurvenverlauf nicht bekannt ist:

Falls der Kurvenverlauf nicht bekannt ist: $= 0.616$ FE



$$\int_a^b f(x)\,dx = \left|\int_a^{x_1} f(x)\,dx\right| + \left|\int_{x_1}^{x_2} f(x)\,dx\right| + \left|\int_{x_2}^{x_3} f(x)\,dx\right| + \left|\int_{x_3}^{b} f(x)\,dx\right|$$

Bp:



$$\int_0^{\pi} \cos(x)\,dx = \left[\sin(x)\right]_0^{\pi} = \sin(\pi) - \sin(0) = 0 - 0 = 0$$

$$A = \left|\int_0^{\pi/2} \cos(x)\,dx\right| + \left|\int_{\frac{\pi}{2}}^{\pi} \cos(x)\,dx\right|$$

$$= \left|\left[\sin(x)\right]_0^{\pi/2}\right| + \left|\left[\sin(x)\right]_{\pi/2}^{\pi}\right| = \left|\sin(\tfrac{\pi}{2}) - \sin(0)\right| + \left|\sin(\pi) - \sin(\tfrac{\pi}{2})\right|$$

$$= |1 - 0| + |0 - 1|$$

$$= |1| + |-1|$$

$$= 1 + 1 = 2$$

Bp: $\int_{-2.5}^{3} (x^3 - 3x^2 - 6x + 8)\,dx$ eingeschl. Flächeninhalt

Teiler von 8 ist z.B. 1 Probe ergibt 1 ist NST $x_1 = 1$

$$x^3 - 3x^2 - 6x + 8 : (x - 1) = \underbrace{x^2 - 2x - 8}$$

$$\begin{array}{l} x^3 - x^2 \\ \hline \quad -2x^2 - 6x \\ \quad -2x^2 + 2x \\ \hline \qquad -8x + 8 \\ \qquad -8x + 8 \\ \hline \qquad \text{"} \quad \text{"} \end{array}$$

$\downarrow$

Weitere NST: $x_2 = 4$

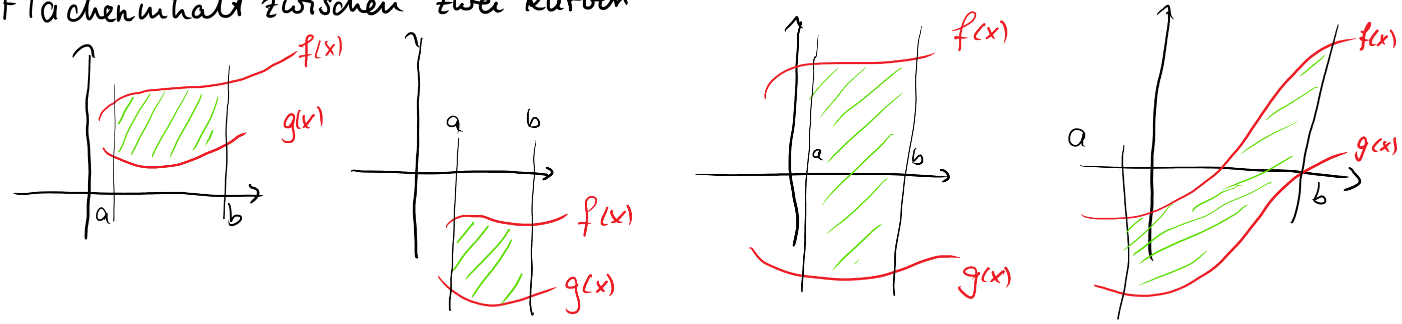
$x_3 = -2$

Integral aufspalten: $\left|\int_{-2.5}^{-2} f(x)\,dx\right| + \left|\int_{-2}^{1} f(x)\,dx\right| + \left|\int_{1}^{3} f(x)\,dx\right|$

Integral aufspalten : $\left| \int_{-2,5} f(x)\,dx \right| + \left| \int_{-2} f(x)\,dx \right| + \left| \int_{1} f(x)\,dx \right|$

Zur Kontrolle : Ergebnis ist 36.855 FE

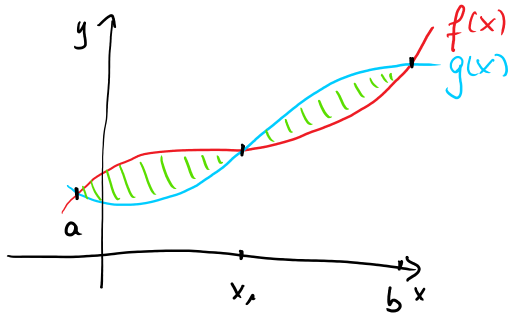
Flächeninhalt zwischen zwei Kurven



Es gilt $f(x) \geq g(x)$ auf $[a,b]$

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b (f(x) - g(x))\,dx$$

Fall :



Eingeschlossene Fläche berechnen

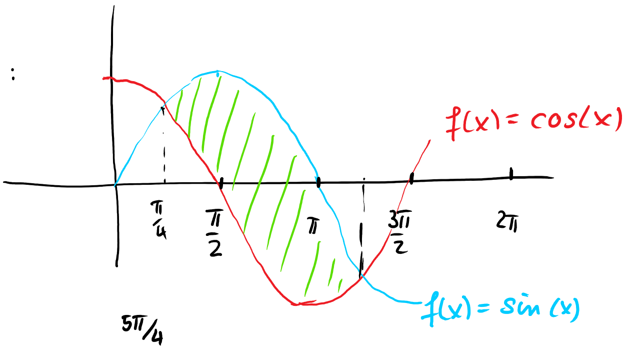
1) Berechnen der Schnittpunkte

auf $[a, x_1]$ $f(x) \geq g(x)$

auf $[x_1, b]$ $g(x) \geq f(x)$

2) $$A = \left| \int_a^{x_1} (f(x) - g(x))\,dx \right| + \left| \int_{x_1}^{b} (g(x) - f(x))\,dx \right|$$

Bp :



Schnittpunkte von $\sin(x)$ und $\cos(x)$

$\cos(x) = \sin(x)$ bei $x_1 = \frac{\pi}{4}$

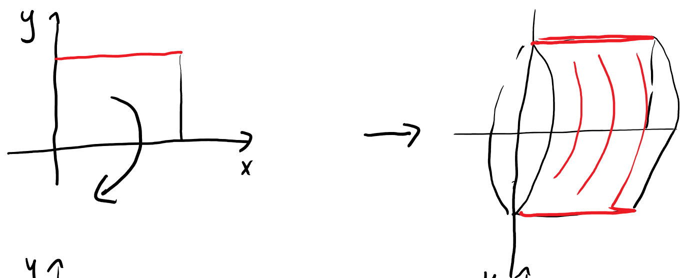
und $x_2 = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$

$$A = \int_{\pi/4}^{5\pi/4} (\sin(x) - \cos(x))\,dx = \Big[-\cos(x) - \sin(x) \Big]_{\pi/4}^{5\pi/4}$$

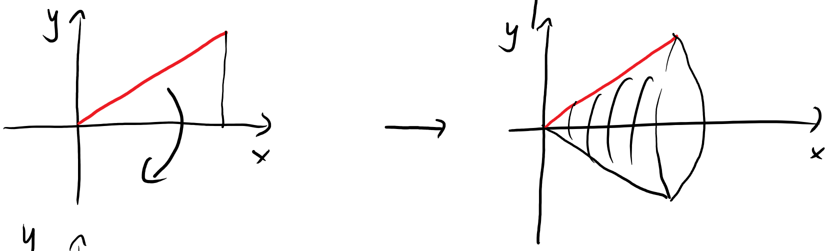
$$= \ldots = \ldots = 2\sqrt{2}\ \text{FE}$$

Rotationskörper

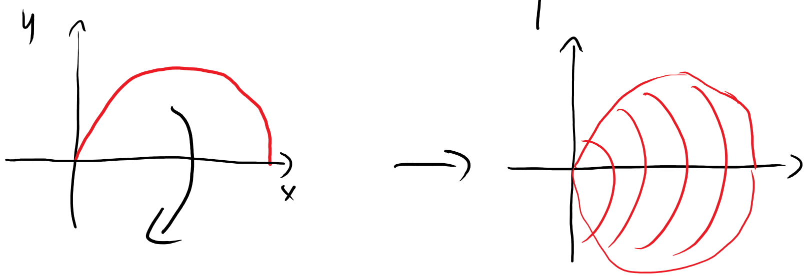
hier: Rotation um die x-Achse



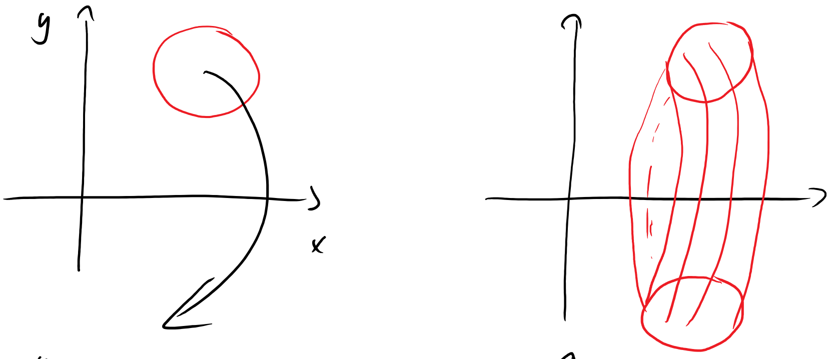
Zylinder



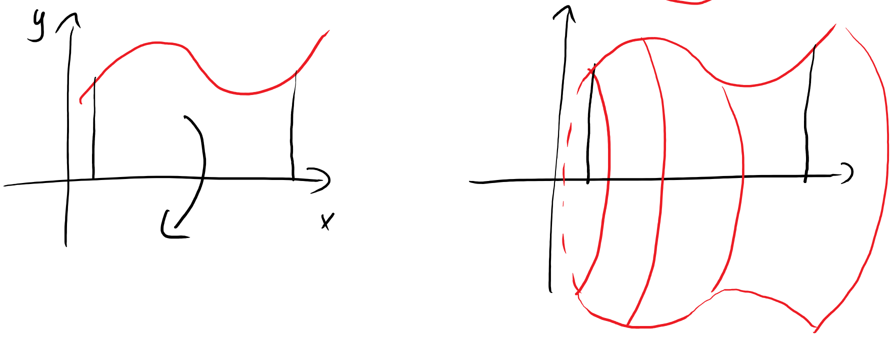
Kegel



Kugel



Donut



"altertümliche Vase"

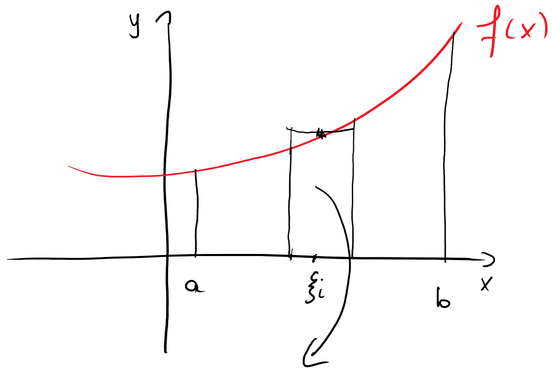
Geg: $y = f(x)$ auf $[a,b]$

Die Fläche rotiere um die x-Achse

Das Volumen des Körpers ist $V = \pi \cdot \int_a^b [f(x)]^2 \cdot dx$

y ↑ $f(x)$

$\breve{a}$



unendlich dünnen Zylinder

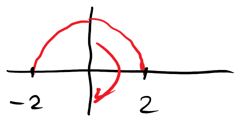
$$\Delta V_i = \underbrace{\pi \cdot \left(f(\xi_i)\right)^2}_{\text{Grundfläche des Zylinders}} \cdot \underbrace{\Delta x_i}_{\text{Höhe des Zylinders}}$$

$$V = \sum_{i=1}^{n} \Delta V_i = \pi \sum_{i=1}^{n} \left[f(\xi_i)\right]^2 \cdot \Delta x_i$$

$$\Big\downarrow \text{ Grenzwert}$$

$$\pi \cdot \int_a^b \left[f(x)\right]^2 \cdot dx$$

Bp: $y = \sqrt{4 - x^2}$ auf $[-2, 2]$ Rotation um x-Achse



$$V = \pi \int_{-2}^{2} \left(\sqrt{4 - x^2}\right)^2 dx = \pi \int_{-2}^{2} (4 - x^2)\, dx$$

$$= \pi \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \pi \left[8 - \frac{8}{3} + 8 - \frac{8}{3}\right]$$

$$= \frac{\pi \cdot 32}{3} = \frac{4}{3} \cdot 2^3 \cdot \pi$$

Kugelvolumen

Hinweis:

Bogenlänge $s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$

Mantelfläche $M_x = 2\pi \int_a^b f(x) \cdot \sqrt{1 + [f'(x)]^2}\, dx$

⋮

Schwerpunkte