Mathematik 28.1.2015

Letzte Vorlesung im WS 2014/15 ☹

Produktintegration

$$\int f \cdot g' \, dx = f \cdot g - \int f' \cdot g \, dx$$

Bp $\int x \cdot \sin x \, dx = x \cdot -\cos x - \int 1 \cdot -\cos x \, dx$

$= -x \cdot \cos x + \int \cos x \, dx$

$= -x \cdot \cos x + \sin x + C, \; C \in \mathbb{R}$

$\boxed{\begin{array}{l} f = x \\ f' = 1 \\ g' = \sin x \\ g = -\cos x \end{array}}$

Bp. $\int \sin x \cdot \cos x \, dx = \sin x \cdot \sin x - \int \cos x \cdot \sin x \, dx$

$f = \sin x$
$f' = \cos x$
$g' = \cos x$
$g = \sin x$

$\Rightarrow \; 2 \cdot \int \sin x \cos x \, dx = \sin^2 x$

$\Rightarrow \; \int \sin x \cdot \cos x \, dx = \frac{1}{2} \sin^2 x + C, \; C \in \mathbb{R}$

Bp $\int 1 \ln x \, dx$ $\quad g'(x) = 1 \quad f(x) = \ln x$

$g(x) = x \quad f'(x) = \frac{1}{x}$

$\int 1 \cdot \ln x = x \cdot \ln x - \int x \cdot \frac{1}{x} \, dx$

$= x \cdot \ln x - \int 1 \, dx$

$= x \cdot \ln x - x = x(\ln x - 1) + C, \; C \in \mathbb{R}$

Integration durch Substitution

"Umkehrung" der Kettenregel

$y' = f'(g(x)) \cdot g'(x)$ (Kettenregel)

Substitutions-Regel : $\int f'(g(x)) \cdot g'(x) \, dx = \int f(z) \, dz \qquad z = g(x)$

Bp: $\int \sqrt{\underbrace{x^2+2}_{z}} \cdot \underbrace{2x \, dx}_{dz} \qquad z = x^2 + 2 \qquad z' = \frac{dz}{dx} = 2x$

$$f(z) = \sqrt{z}$$

$$= \int \sqrt{z}\, dz$$

mit $\frac{dz}{dx} = 2x$ bzw. $dz = 2x\,dx$

$$= \int z^{\frac{1}{2}}\, dz = \frac{2}{3} z^{\frac{3}{2}}$$

$$\underset{\text{Rücksubstitution}}{\Rightarrow} \frac{2}{3}(x^2+2)^{\frac{3}{2}} + C, \quad C \in \mathbb{R}$$

Allgemeines Vorgehen bei der Integration durch Substitution

1) Aufstellen der Substitutionsgleichung $z = g(x)$

2) Berechnen von $z' = \frac{dz}{dx} \Rightarrow dz = g'(x)\,dx$

3) Anwenden der Formel: $\int f(g(x)) \cdot g'(x)\,dx = \int f(z)\,dz$

Bp: 1) $\int \frac{1}{2x+3}\, dx \qquad z = 2x+3 \qquad \frac{dz}{dx} = 2 \qquad dz = 2\,dx$

$$\frac{1}{2}\int \frac{1}{z}\, \frac{dz}{2\,dx} = \frac{1}{2}\int \frac{1}{z}\, dz = \frac{1}{2}\ln|z| + C$$

$$\underset{\text{Rücksubst.}}{=} \frac{1}{2}\ln|2x+3| + C$$

2) $\int x \cdot e^{-x^2}\, dx \qquad z = -x^2 \qquad \frac{dz}{dx} = -2x \qquad dz = -2x\,dx$

$$= \left(-\frac{1}{2}\right)\int e^{z}\, \frac{dz}{-2x\,dx} = -\frac{1}{2}e^{z} \underset{\text{Rücksubst.}}{=} -\frac{1}{2}e^{-x^2} + C, \quad C \in \mathbb{R}$$

3) $\boxed{\int \frac{\ln x}{x}\, dx} = \int (\ln x) \cdot \underbrace{\frac{1}{x}\, dx}_{dz} \qquad z = \ln x \qquad \frac{dz}{dx} = \frac{1}{x} \qquad dz = \frac{1}{x}\,dx$

$$\int z\, dz = \frac{1}{2}z^2 \underset{\text{Rücksubst.}}{=} \boxed{\frac{1}{2}(\ln x)^2 + C}, \quad C \in \mathbb{R}$$

4) $\int \frac{\sin x \cdot \cos x}{1 + \sin^2 x}\, dx \qquad z = 1 + \sin^2 x \qquad \frac{dz}{dx} = 2\sin x \cdot \cos x$

$dz = 2\sin x \cdot \cos x\, dx$

$$= \frac{1}{2}\int \frac{1}{z}\, dz$$

$$= \frac{1}{2}\ln|z| \underset{\text{Rücksubst.}}{=} \frac{1}{2}\ln|1+\sin^2 x| + C, \quad C \in \mathbb{R}$$

Nur kurz erwähnt:

Differentialrechnung → Integralrechnung

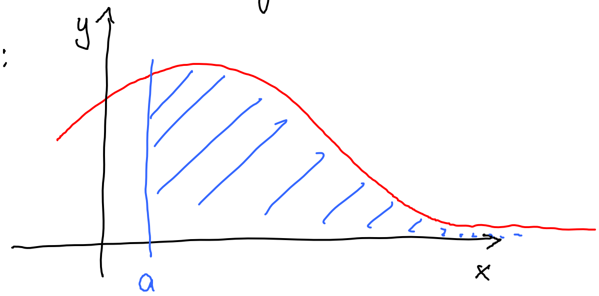
Produktregel → Produktintegration

Kettenregel → Integration durch Substitution

Quotientenregel → Integration mit Hilfe der Partialbruchzerlegung

Uneigentliche Integrale

Fall:



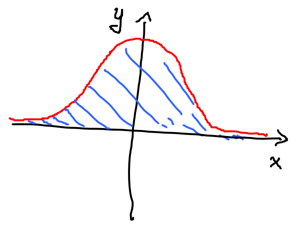
$$\int_a^{\infty} f(x)\,dx$$

$$\lim_{b_n \to \infty} \int_a^{b_n} f(x)\,dx = \int_a^{\infty} f(x)\,dx$$

bzw.

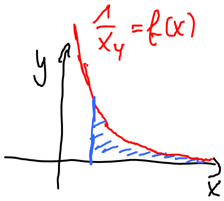
$$\lim_{a_n \to -\infty} \int_{a_n}^{b} f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx$$

$$\lim_{\substack{b_n \to \infty \\ a_n \to -\infty}} \int_{a_n}^{b_n} f(x)\,dx = \int_{-\infty}^{+\infty} f(x)\,dx$$



Hinweis → SS
Standardnormalverteilung
Fläche unter der Gauß-Glocke $= 1$

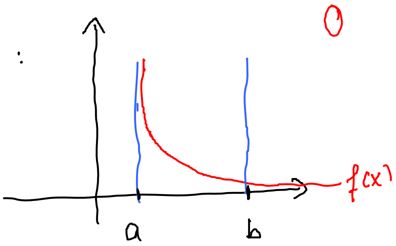
Bp: $\displaystyle\int_1^{\infty} \frac{1}{x^4}\,dx$



$$\int_1^b \frac{1}{x^4}\,dx = \left[-\frac{1}{3}x^{-3}\right]_1^b = \frac{b^{-3}}{-3} - \frac{1^{-3}}{-3} = -\frac{1}{3b^3} + \frac{1}{3}$$

$$\lim_{b \to \infty}\left(\underbrace{-\frac{1}{3b^3}}_{\to 0} + \frac{1}{3}\right) = \frac{1}{3}$$

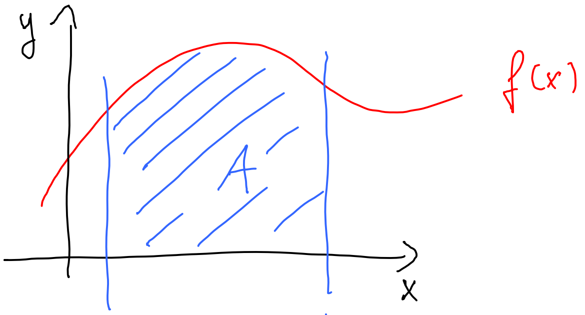
Fall:



Der Integrand ist nicht beschränkt, d.h.
f ist stetig auf $]a, b]$ mit $\lim_{x \to a} f(x) = \pm\infty$
Polstelle bei $x = a$

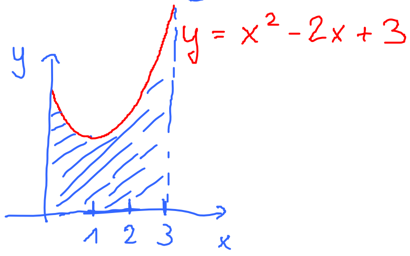
Anwendungen der Integralrechnung

1) Flächeninhalt



$$A = \int_a^b f(x)\,dx \qquad f(x) \geq 0 \text{ auf } [a,b]$$

Bp

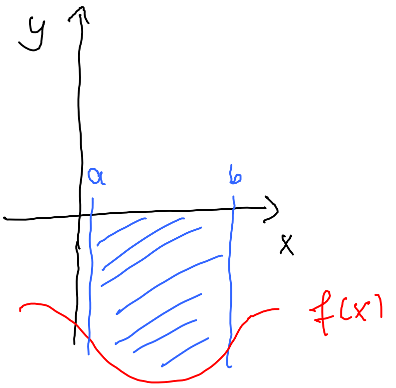


$$A = \int_0^3 x^2 - 2x + 3\,dx$$

$$= \left[\frac{x^3}{3} - \frac{2x^2}{2} + 3x\right]_0^3 = \frac{27}{3} - \frac{18}{2} + 9 - 0$$

$$= 9 - 9 + 9 = 9 \text{ FE}$$

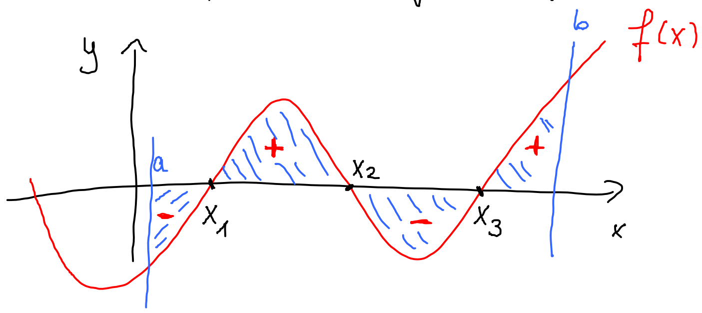
Fall:



$$A = \left| \int_a^b f(x)\,dx \right| = -\int_a^b f(x)\,dx$$

$$f(x) \leq 0 \text{ auf } [a,b]$$

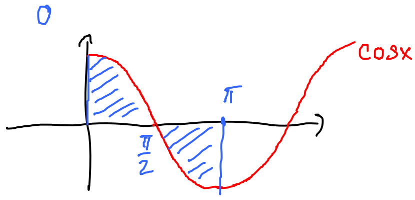
Fall:



Als "Fläche"

$$\int_a^b f(x)\,dx = \left|\int_a^{x_1} f(x)dx\right| + \left|\int_{x_1}^{x_2} f(x)dx\right| + \left|\int_{x_2}^{x_3} f(x)\,dx\right| + \left|\int_{x_3}^{b} f(x)\,dx\right|$$

Bp: $\int_0^{\pi} \cos x\,dx = \left[\sin x\right]_0^{\pi} = \sin\pi - \sin 0 = 0 - 0 = 0$



Tatsächliche Fläche: $\left|\int_0^{\pi/2} \cos x\,dx\right| + \left|\int_{\pi/2}^{\pi} \cos x\,dx\right|$

$$= \left[\sin x\right]_0^{\pi/2} + \left[\sin x\right]_{\pi/2}^{\pi}$$

$$= \left|\sin\tfrac{\pi}{2} - \sin 0\right| + \left|\sin\pi - \sin\tfrac{\pi}{2}\right|$$
$$= |1 - 0| + |0 - 1|$$
$$= 1 + |-1| = 1 + 1 = 2\,FE$$

Bp: Berechnen Sie den Flächeninhalt, den $f(x) = x^3 - 3x^2 - 6x + 8$ mit der x-Achse auf $[-2{,}5;\ 3]$ einschließt!

Nullstellenbestimmung: 1. NST Erraten (Teiler des Absolutgliedes durchprobieren)

$x_1 = 1$

$$\begin{array}{l} (x^3 - 3x^2 - 6x + 8) : (x-1) = \underbrace{x^2 - 2x - 8} \\ \underline{x^3 - x^2} \\ \quad -2x^2 - 6x \\ \quad \underline{-2x^2 + 2x} \\ \qquad -8x + 8 \\ \qquad -8x + 8 \end{array}$$

↓ p,q-Formel

$x_2 = 4$

$x_3 = -2$

$$\int_{-2{,}5}^{3} f(x)\,dx = \underbrace{\left|\int_{-2{,}5}^{-2} f(x)\,dx\right|}_{I_1} + \underbrace{\left|\int_{-2}^{1} f(x)\,dx\right|}_{I_2} + \underbrace{\left|\int_{1}^{3} f(x)\,dx\right|}_{I_3}$$

$I_1 = \left[\frac{x^4}{4} - \frac{3x^3}{3} - \frac{6x^2}{2} + 8x\right]_{-2{,}5}^{-2}$

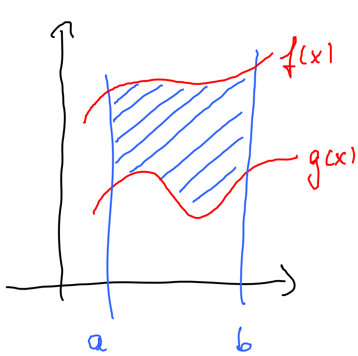
I_2 ↓ entsprechend

I_3 ↓ entsprechend

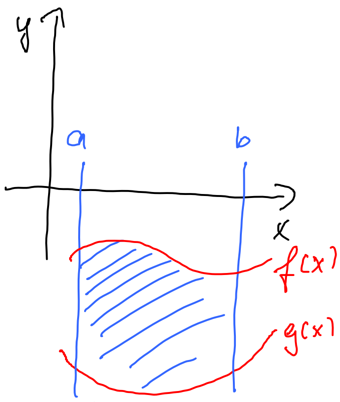
$$= |-2{,}635| + |20{,}25| + |-14|$$

Die Gesamtfläche hat den Inhalt: $36{,}885\,FE$

Flächeninhalt zwischen zwei Kurven

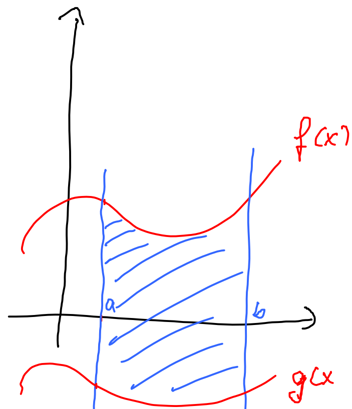


$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b (f(x) - g(x))\,dx$$



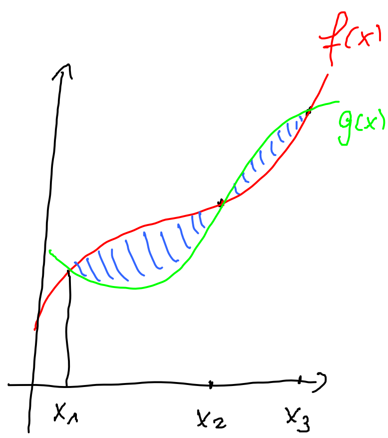
$$A = \int_a^b (f(x) - g(x))\,dx$$

!



$$A = \int_a^b (f(x) - g(x))\,dx$$

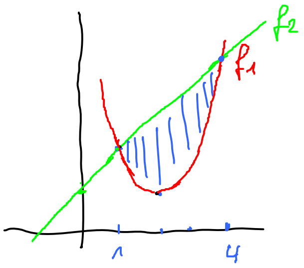
!



Fläche die eingeschlossen wird von f(x) und g(x)

$$\int_{x_1}^{x_2} f(x) - g(x)\,dx + \int_{x_2}^{x_3} g(x) - f(x)\,dx$$

Bp: 1) Berechnen Sie die Fläche, die von $f_1(x) = x^2 - 4x + 5$ und $f_2(x) = x+1$ eingeschlossen wird



$$\int_1^4 (x+1) - (x^2 - 4x + 5)\, dx$$

$$\frac{27}{6} = 4{,}5 \text{ FE}$$

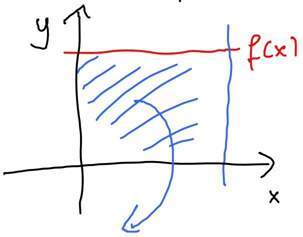
Die Grenzen sind zu berechnen aus den Schnittpunkten von f_1 und f_2:

$x_1 = 1 \quad x_2 = 4$

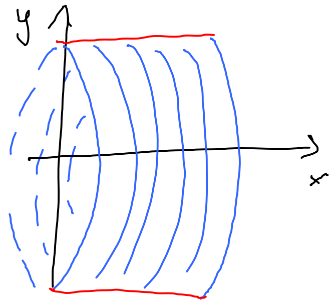
2) (zu Hause)

Fläche zwischen $y = x^2$ und $y = \sqrt{x}$ auf $[0,2]$

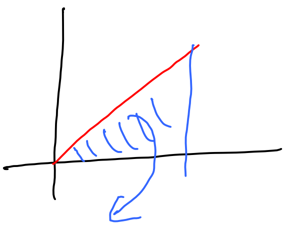
Rotationskörper hier: um die x-Achse



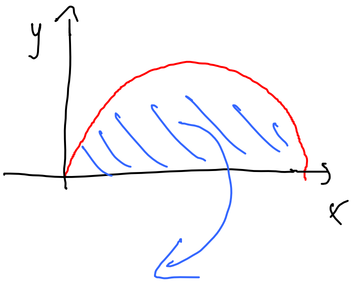
→



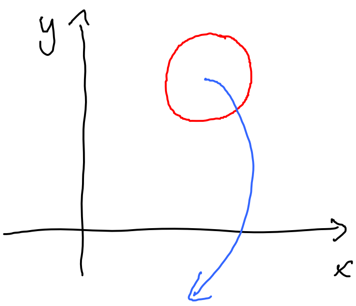
Zylinder



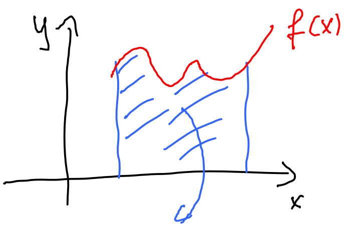
→ Kegel



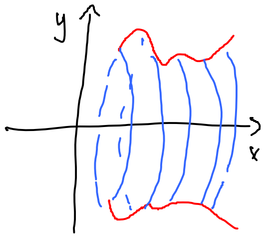
→ Kugel



→ Kreisring, Donut

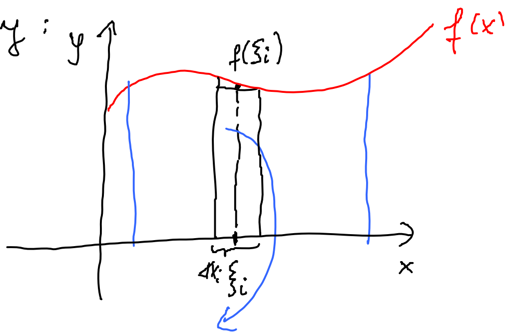


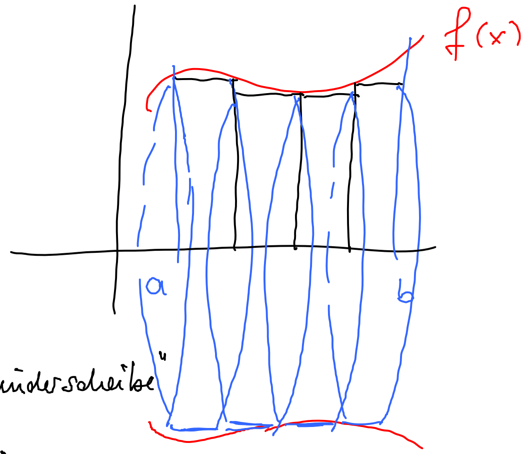
→



"Körper"

Herleitung:





bei Rotation entsteht eine "Zylinderscheibe"

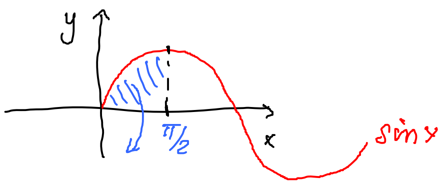
$$\Delta V_i = \pi \cdot \left[f(\xi_i)\right]^2 \cdot \Delta x_i$$

Summe von Zylinderscheiben

$$V = \sum_{i=1}^{n} \Delta V_i = \pi \sum_{i=1}^{n} \left[f(\xi_i)\right]^2 \underbrace{(x_i - x_{i-1})}_{\Delta x_i}$$

$$= \lim_{\substack{n\to\infty \\ \Delta x_i \to 0}} \sum_{i=1}^{n} \Delta V_i = \pi \int_a^b \left[f(x)\right]^2 dx$$

Bp. $y = \sin x$ rotiere um die x-Achse auf $\left[0, \frac{\pi}{2}\right]$



$$V = \pi \int_0^{\pi/2} (\sin x)^2 \, dx$$

Das Integral wird mit der partiellen Integration gelöst $f(x) = \sin x$ ---
$g'(x) = \sin x$

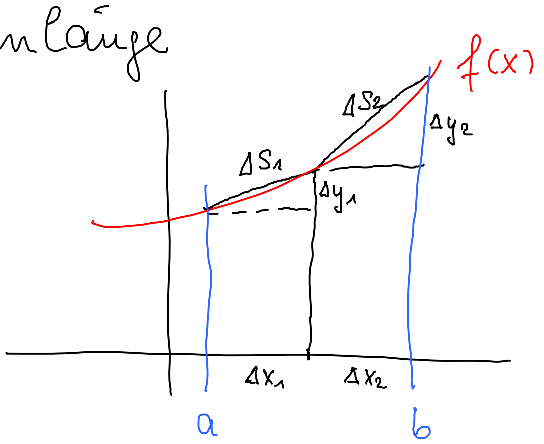
$$V = \pi \left[\frac{1}{2}(x - \sin x \cdot \cos x)\right]_0^{\pi/2}$$

$$= \frac{\pi^2}{4} \text{ VE}$$

Zu Hause: Volumen einer Kugel

$$V = \pi \int_{-2}^{2} \Big(\underbrace{\sqrt{4-x^2}}_{\text{Halbkreis}}\Big)^2 dx$$

Bogenlänge



Länge des Kurvenstücks auf $[a,b]$

$$S = \int_a^b \sqrt{1 + (f'(x))^2} \, dx$$