Vorlesung Mathematik
27.1.2016

"die letzte im WS 15/16"

Sonderfälle zur partiellen Integration:

1) $\int \ln x \, dx = \int 1 \cdot \ln x \, dx$ $\qquad f(x) = \ln x \quad g'(x) = 1$
$\qquad f'(x) = \frac{1}{x} \quad g(x) = x$

$$= \ln x \cdot x - \int \frac{1}{x} \cdot x \, dx$$
$$= x \cdot \ln x - \int 1 \, dx$$
$$= x \cdot \ln x - x = x(\ln x - 1) + c, \; c \in \mathbb{R}$$

2) $\int \sin(x) \cdot \cos(x) \, dx$ $\qquad f(x) = \sin(x) \quad g'(x) = \cos(x)$
$\qquad f'(x) = \cos(x) \quad g(x) = \sin(x)$

$$\underbrace{\int \overset{f(x)}{\sin(x)} \cdot \overset{g'(x)}{\cos(x)} \, dx} = \sin(x) \cdot \sin(x) - \underbrace{\int \cos(x) \cdot \sin(x) \, dx}$$

$$2 \int \sin(x) \cdot \cos(x) \, dx = \sin^2(x)$$

$$\int \sin(x)\cdot\cos(x)\,dx = \frac{1}{2}\sin^2(x) + C,\ C\in\mathbb{R}$$

Integration durch Substitution

Bp: $\int \underbrace{\sqrt{x^2+2}}_{z}\ \underbrace{\boxed{\cdot 2x\ dx}}_{dz}$

NR: $\left(\sqrt{x^2+2}\right)' = \left((x^2+2)^{\frac{1}{2}}\right)' = \frac{1}{2}(x^2+2)^{-\frac{1}{2}}\cdot \underbrace{2x}_{\text{innere Abl.}}$

Substitution: $z = x^2+2$

$$z' = \frac{dz}{dx} = 2x$$

$$\Leftrightarrow \boxed{dz = 2x\cdot dx}$$

$$\int \sqrt{z}\,dz = \int z^{\frac{1}{2}}\,dz = \frac{2}{3}\cdot z^{\frac{3}{2}} \underset{\text{Rücksubst.}}{=} \frac{2}{3}(x^2+2)^{\frac{3}{2}} + C,\ C\in\mathbb{R}$$

Regel: $\int f(g(x))\cdot g'(x)\,dx = \int f(z)\,dz$ mit $z = g(x)$

Bp: 1) $\int \frac{1}{2x+3}\,dx$ Substitution: $z = 2x+3$

$\frac{dz}{dx} = 2 \quad dz = 2dx$

$$\frac{1}{2}\int \frac{1}{z}\ \underset{2dx}{dz}$$

$$= \frac{1}{2}\int \frac{1}{z}\,dz = \frac{1}{2}\ln|z| \underset{\text{Rücksubst.}}{=} \frac{1}{2}\ln|2x+3| + C,\ C\in\mathbb{R}$$

2) $\int x\cdot e^{-x^2}\,dx$ Substitution: $z = -x^2$

$z' = \frac{dz}{dx} = -2x$

$dz = -2x\,dx$

$$\int e^z\ x\cdot dx$$

$$-\frac{1}{2}\int e^z\ \underset{-2x\,dx}{dz}$$

$$-\frac{1}{2}\int e^z\,dz = -\frac{1}{2}e^z \underset{\text{Rücksubst.}}{=} -\frac{1}{2}e^{-x^2} + C,\ C\in\mathbb{R}$$

3) $\int \frac{\sin(x)\cdot\cos(x)}{1+\sin^2(x)}\,dx$ $z = 1+\sin^2(x)$

$z' = \frac{dz}{dx} = 2\sin(x)\cdot\cos(x)$

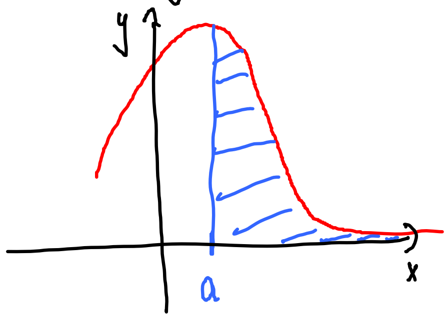
$$dz = 2\sin(x)\cdot\cos(x)\cdot dx$$

$$= \frac{1}{2}\int \frac{1}{z}\,dz$$

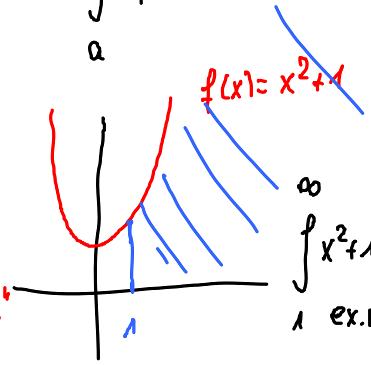
$$= \frac{1}{2}\ln|z| \underset{RS}{=} \frac{1}{2}\ln\left|1+\sin^2(x)\right| + C,\ C\in\mathbb{R}$$

Achtung: Bei bestimmten Integralen müssen die Grenzen auch substituiert werden, am Ende auch rücksubstituiert!
s. Üb.bl. 5 bzw. Übungen

Uneigentliche Integrale



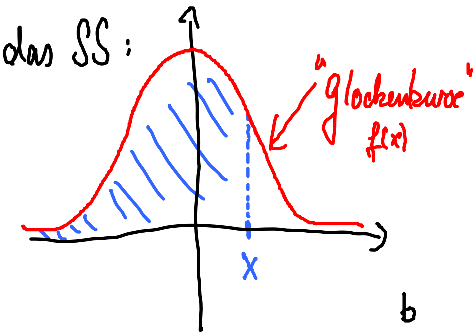
$$\int_a^\infty f(x)\,dx$$



$$\int_1^\infty x^2+1$$ ex. nicht

Hinweis auf das SS:

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1$$



Bp.: $\int_0^\infty e^{-x}\,dx$

$$\lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-x}\right]_0^b$$

$$= \lim_{b\to\infty}\left(-e^{-b}+e^0\right)$$

$$= \lim_{b\to\infty}\left(\underbrace{-\frac{1}{e^b}}_{\to 0}+1\right)$$

$$= 1$$

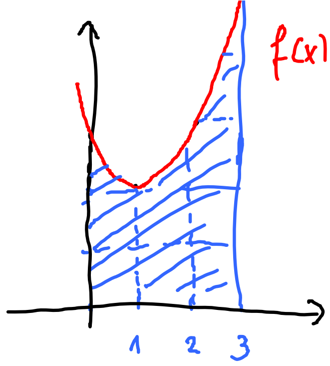
$$\int e^{-x}\,dx \qquad z=-x$$

$$\frac{dz}{dx} = -1 \qquad dz = -dx$$

$$-\int e^z\,dz = -e^z \underset{RS}{=} -e^{-x}$$

Anwendungen der Integralrechnung

1) Flächeninhalt



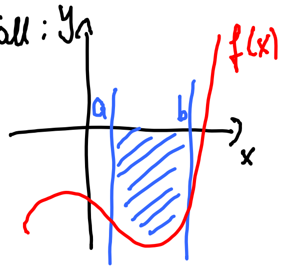
$f(x) = x^2 - 2x + 3$

$$\int_0^3 \underbrace{x^2-2x+3}\,dx = \left[\frac{1}{3}x^3 - \frac{\cancel{2}x^2}{\cancel{2}} + 3x\right]_0^3$$

$$= \left(\frac{1}{3}\cdot 3^3 - 3^2 + 3\cdot 3 - 0\right)$$

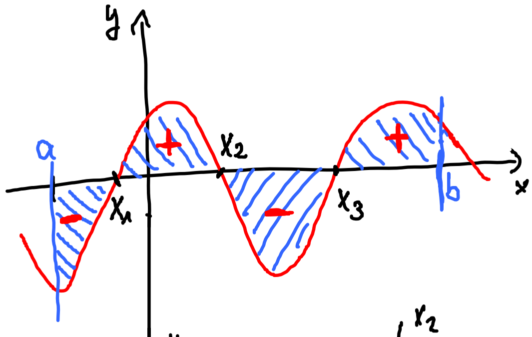
$$= 3^2 - 3^2 + 3^2 = 9 \text{ FE}$$

Folgendes Fall:



$$A = \left|\int_a^b f(x)\,dx\right| = -\int_a^b f(x)\,dx$$

Falls folgende Situation vorliegt:



$$= \left|\int_a^{x_2} f(x)\,dx\right| + \left|\int_{x_1}^{x_2} f(x)\,dx\right| + \left|\int_{x_2}^{x_3} f(x)\,dx\right| + \left|\int_{x_3}^{b} f(x)\,dx\right|$$

Bp: $\int_{-2,5}^{3} (x^3 - 3x^2 - 6x + 8)\,dx$ Gesucht: der Flächeninhalt!

NST berechnen : 1. NST durch Erraten $x_1 = 1$

Weitere NST : Polynomdivision

$$\begin{array}{l}
x^3 - 3x^2 - 6x + 8 : (x-1) = \underline{x^2 - 2x - 8} \\
-\ x^3 - x^2 \\
\hline
0 \quad -2x^2 - 6x \\
 -2x^2 + 2x \\
\hline
 0 \ -8x + 8 \\
 -8x + 8
\end{array}$$

$$x^2 - 2x - 8 = 0$$

nach a,b,c-Formel

$$x_{2,3} = 1 \pm \sqrt{9} \qquad\qquad x_{2,3} = \frac{2 \pm \sqrt{4+32}}{2} = \frac{2 \pm 6}{2}$$

$$x_2 = 4 \qquad\qquad x_2 = 4$$

$$x_3 = -2 \qquad\qquad x_3 = -2$$

Folgende NST: $x_1 = 1$, $x_2 = 4$, $x_3 = -2$

$$\int_{-2,5}^{3} f(x)\,dx = \left|\int_{-2,5}^{-2} f(x)\,dx\right| + \left|\int_{-2}^{1} f(x)\,dx\right| + \left|\int_{1}^{3} f(x)\,dx\right|$$

$$\overset{-}{I_1} \qquad \overset{+}{I_2} \qquad -I_3$$

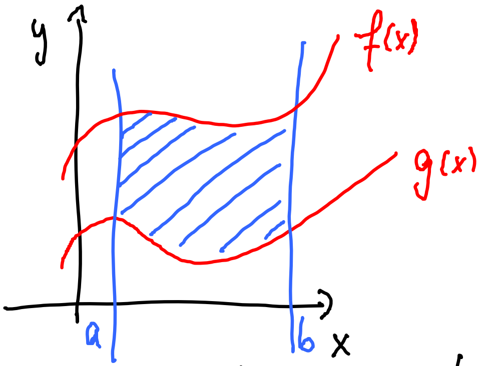
$f(x) = x^3 - 3x^2 - 6x + 8$

Stammfunktion: $\left[\frac{1}{4}x^4 - x^3 - 3x^2 + 8x\right]$

$I_1 = -2{,}635 \qquad I_2 = 20{,}25 \qquad I_3 = -14$

$$A = |I_1| + |I_2| + |I_3| = 36{,}88\,FE$$

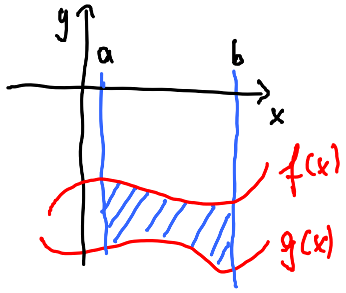
Fläche zwischen zwei Kurven



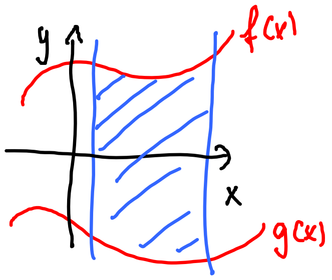
$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b \big(f(x) - g(x)\big)\,dx$$

Immer anzuwenden, wenn $f(x) \geq g(x)$ auf $[a,b]$

$f(x) \geq g(x)$

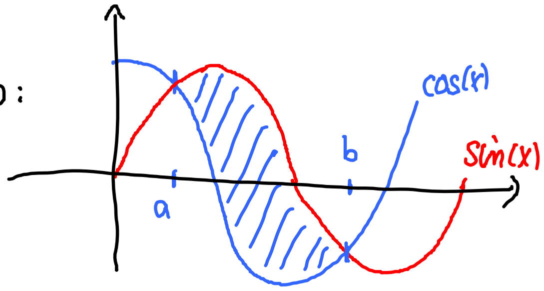


$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$



$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Bp:



$$\int_a^b (\sin(x) - \cos(x))\,dx$$

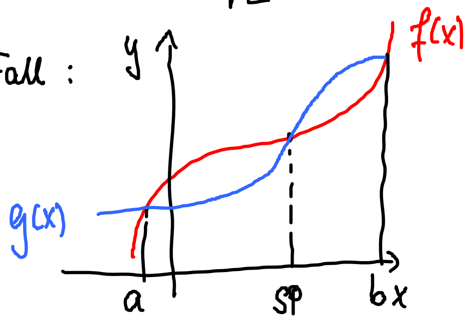
NP: $\sin(x) = \cos(x) \Leftrightarrow x = \frac{\pi}{4}$ bzw $\frac{5\pi}{4}$

$$\int_{\pi/4}^{5\pi/4} (\sin(x) - \cos(x))\,dx = \Big[-\cos(x) - \sin(x)\Big]_{\pi/4}^{5\pi/4}$$

$$= -\left(-\tfrac{1}{2}\sqrt{2}\right) - \left(-\tfrac{1}{2}\sqrt{2}\right) + \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2}$$

$$= 2\sqrt{2} \text{ FE}$$

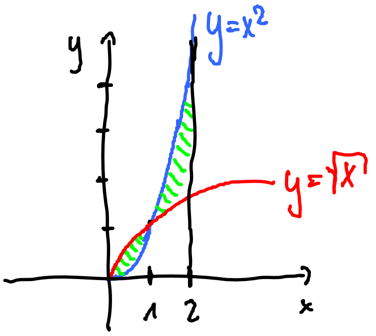
Weiterer Fall:



$f(x) \geq g(x)$ $\quad$ $g(x) \geq f(x)$

Notwendig: Berechnung der Schnittpunkte von $f(x)$ und $g(x)$

Bp:

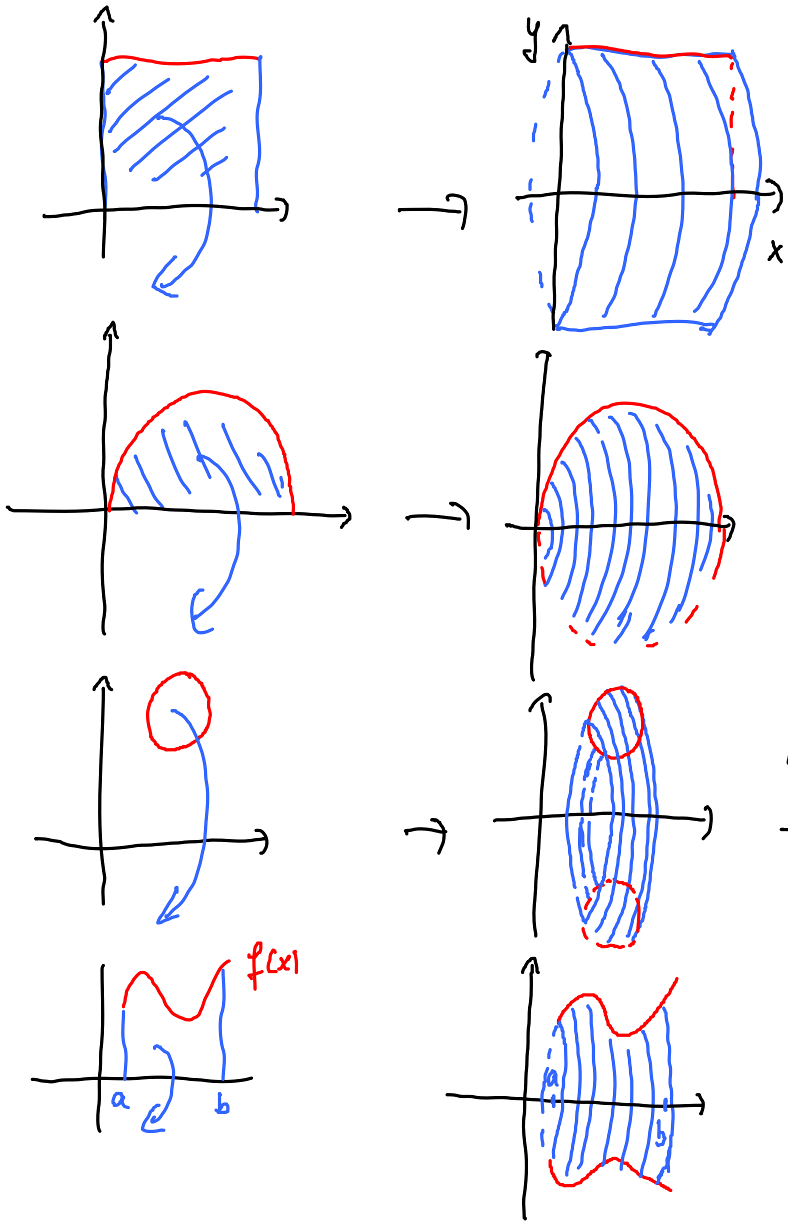


$$\int_0^1 (\sqrt{x} - x^2)\,dx + \int_1^2 (x^2 - \sqrt{x})\,dx$$

$$= \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3} - \frac{2}{3}x^{3/2}\right]_1^2$$

$$= \ldots = \ldots = \frac{10 - 4\sqrt{2}}{3} \approx 1{,}447 \text{ FE}$$

Rotationskörper (Rotation um die x-Achse)

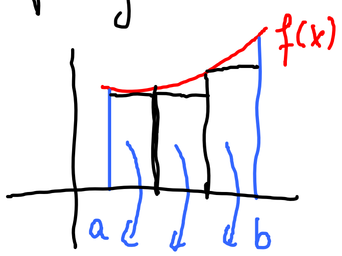


Rechteck
→ Zylinder

Halbkreis
→ Kugel

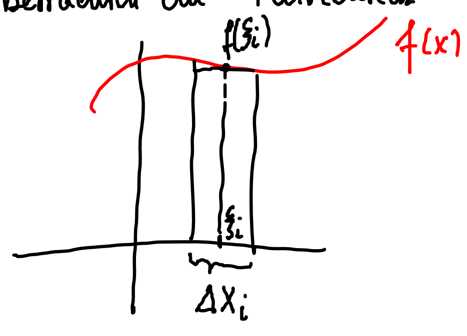
Kreis
→ Kreisring
"Torus"

Hinführung zur Formel:



z.B. 3 Teilintervalle
mit 3 Rechtecken, die um die x-Achse rotieren

Wir betrachten ein "Teilrechteck"



Teilzylindervolumen
mit $r = f(\xi_i)$
$h = \Delta x_i$

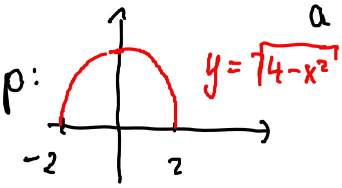
$\pi \cdot r^2 \cdot h$

$$V = \sum_{i=1}^{n} \Delta V_i = \pi \sum_{i=1}^{n} \left[f(\xi_i)\right]^2 \cdot \Delta x_i \qquad \text{damit } V_i = \pi \left[f(\xi_i)\right]^2 \cdot \Delta x_i$$

für $n \to \infty$ und $\Delta x_i \to 0$

$$V = \pi \int_a^b \left[f(x)\right]^2 \cdot dx$$

Bp:



Rotationsvolumen bei Drehung um die x-Achse:

$$V = \pi \int_{-2}^{2} \left(\sqrt{4-x^2}\right)^2 dx$$

$$= \pi \int_{-2}^{2} (4-x^2)\, dx$$

$$= \pi \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

$$= \pi \left(8 - \frac{8}{3} + 8 - \frac{3}{3}\right) = \pi \cdot \frac{32}{3} \text{ VE}$$

$$= \frac{4}{3} \cdot \pi \cdot 2^3$$

(Allg. Kugelvolumen: $\frac{4}{3}\pi \cdot r^3$)