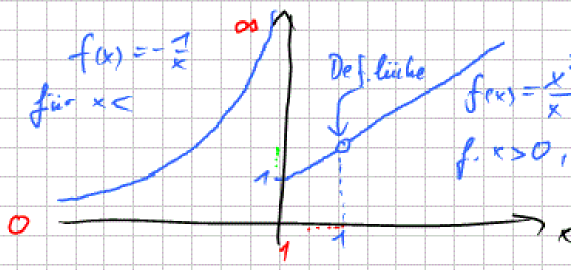
V MA1 – 9.11.2016

Grenzwert Funktionen



$f(x) = \frac{x^2-1}{x-1} = \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = \underline{x+1} \quad \forall x \neq 1$

f. $x>0,\ x\neq 1$

$\lim_{x\to-\infty} f(x) = 0$, denn für $(x_n) \xrightarrow{n\to\infty} -\infty$ gilt $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty}\left(-\frac{1}{x_n}\right) = 0$

$\lim_{x\to 0-} f(x) = +\infty$, denn für $x_n = -\frac{1}{n} \xrightarrow{n\to\infty} 0,\ x_n<0$, gilt $\lim_{n\to\infty}\left(-\frac{1}{\left(-\frac{1}{n}\right)}\right) = \lim_{n\to\infty} n = +\infty$

$\lim_{x\to 0+} f(x) = 1$, denn $f(0) = \frac{0^2-1}{0-1} = \underline{\underline{+1}}$

$\lim_{x\to 1} f(x) = 2$, denn für $(x_n) \to 1,\ x_n \neq 1$

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{x_n^2-1}{x_n-1} = \lim_{n\to\infty} x_n+1 = 1+1 = \underline{\underline{2}}$$

Differenzieren

Ü $\boxed{f(x) = x^2}$

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$
$$= \lim_{h\to 0} \frac{(x_0+h)^2 - x_0^2}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{x_0^2} + 2\cancel{h}x_0 + h^2 - \cancel{x_0^2}}{\cancel{h}}$$
$$= \lim_{h\to 0} \frac{2x_0 + h}{1} = 2x_0$$

x	$f(x)$
x	x^2
x_0	x_0^2
A	A^2
x_0+h	$(x_0+h)^2$

Binomischer Satz

$f(x) = x^n$: $\quad f'(x_0) = \lim_{h\to 0} \frac{(x_0+h)^n - x_0^n}{h} = \frac{\cancel{x_0^n} + n\cancel{h}x_0^{n-1} + \ldots - \cancel{x_0^n}}{\cancel{h}}$

$$= \lim_{h\to 0} \frac{n x_0^{n-1} + \ldots}{1} = n x_0^{n-1}$$

$$f(x) = x^n \Rightarrow \boxed{f'(x) = n x^{n-1}}$$

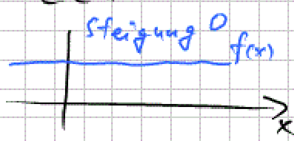
Beispiel Quotientenregel

$$f(x) = \frac{x^2}{x+1}$$

$$u(x) = x^2, \quad u'(x) = 2x$$
$$v(x) = x+1, \quad v'(x) = 1$$

$$f'(x) = \frac{(x+1)2x - 1 \cdot x^2}{(x+1)^2}$$
$$= \frac{2x^2 + 2x - x^2}{(x+1)^2}$$
$$= \frac{x^2 + 2x}{(x+1)^2}$$

$$f(x) = c, \quad c \in \mathbb{R}$$
$$f'(x) = 0$$



Beispiel Kettenregel

$$h(x) = \left(\frac{x^2}{x+1}\right)^2$$

$$h'(x) = g'(f(x)) \cdot f'(x)$$
$$= 2\,\frac{x^2}{x+1} \cdot \frac{x^2+2x}{(x+1)^2}$$
$$= \underline{\underline{\frac{2x^4 + 4x^3}{(x+1)^3}}}$$

N.R.

$$f(x) = \frac{x^2}{x+1} = u$$
$$g(u) = u^2$$
$$g(f(x)) = g\left(\frac{x^2}{x+1}\right) = \left(\frac{x^2}{x+1}\right)^2 = \underline{\underline{h(x)}}$$
$$g'(u) = 2u$$
$$f'(x) = \frac{x^2+2x}{(x+1)^2} \quad \text{(s.o.)}$$

Übung

a) $f(x) = \dfrac{x^3}{(x+1)^2}$

$$f'(x) = \frac{(x+1)^{\cancel{2}\,1}\,3x^2 - 2\cancel{(x+1)}\,x^3}{(x+1)^{\cancel{4}\,3}}$$
$$= \frac{3x^3 + 3x^2 - 2x^3}{(x+1)^3}$$
$$= \underline{\underline{\frac{x^3 + 3x^2}{(x+1)^3}}}$$

Quotienregel

$$u(x) = x^3 \Rightarrow u'(x) = 3x^2$$
$$v(x) = (x+1)^2 \Rightarrow v'(x) = 2(x+1)$$
$$= x^2 + 2x + 2$$

Kettenregel

$$v'(x) = 2x + 2$$
$$= 2(x+1)$$

b) $g(x) = \sin(x^3)$

$$g'(x) = \cos(x^3) \cdot 3x^2$$

Kettenregel

$$g(x) = g_1(g_2(x))$$
$$g_2(x) = x^3 \qquad g_2' = 3x^2$$
$$g_1(x) = \sin(x) \qquad g_1' = \cos(x)$$

c) $h(x) = e^{\sin(x^3)}$

Kettenregel

$$g(u) = e^u \qquad g'(u) = e^u$$

$h'(x) = \underbrace{e^{\sin(x^2)}}_{g'(u)} \cdot \underbrace{\cos(x^2) \cdot 2x}_{f'(x)}$ $\quad\Big|\quad$ $f(x) = \sin(x^2) \xRightarrow[\text{Kettenregel}]{} f'(x) = \cos(x^2) \cdot 2x$

Ableitung Betragsfunktion

$$f(x) = |x| = \begin{cases} x & f \; x > 0 \\ -x & f \; x \leq 0 \end{cases}$$

Fall 1 : $x > 0$: $f'(x) = 1$

Fall 2 : $x < 0$: $f'(x) = -1$

$f'(0)$ existiert nicht, weil links- und rechtsseitiger Grenzwert verschieden (-1 bzw $+1$)

$$f(x) = |(x+1)^3|$$

Fall 1: $x+1 > 0$: $f(x) = (x+1)^3 \Rightarrow f'(x) = 3(x+1)^2 \Rightarrow f'(-1) = 0$

Fall 2: $x+1 < 0$: $f(x) = -(x+1)^3 \Rightarrow f'(x) = -3(x+1)^2 \Rightarrow f'(-1) = -0 = 0$

$f'(0)$ ist links wie rechts gleich, also existiert $f'(0)$ auch insgesamt