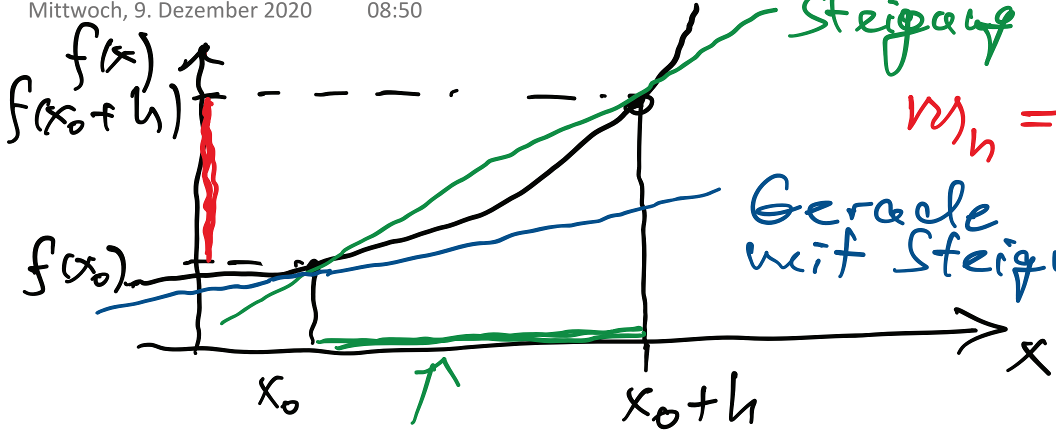
Diff 1

Mittwoch, 9. Dezember 2020 08:50



Gerade mit Steigung m_h

$$m_h = \frac{f(x_0+h) - f(x_0)}{x_0 + h - x_0}$$

Gerade mit Steigung $f'(x_0)$

$x_0 + h - x_0 = h$

$$\lim_{h \to 0} m_h = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$= f'(x_0)$$

Bsp. $f(x) = x^2$, Was ist $f'(x)$?

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \qquad \left(\tfrac{0}{0}\text{-Situation}\right)$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2hx + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} 2x + h = \underline{\underline{2x}}$$

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allg. Regel: $f(x) = x^c$

$$\Rightarrow f'(x) = c\,x^{c-1} \quad \forall c \in \mathbb{R}$$

$\boxed{c \neq 0}$

z.B. $f(x) = x^{3.5}$

$f'(x) = 3.5\,x^{2.5}$

Diff Quotient

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Bsp: $f(x) = \dfrac{x^2}{x+1}$ Was ist $f'(x)$?

$$f'(x) = \frac{\overbrace{(x+1)}^{v}\cdot\overbrace{2x}^{u'} - \overbrace{x^2}^{u}\cdot\overbrace{1}^{v'}}{(x+1)^2}$$

$$= \frac{2x^2+2x-x^2}{(x+1)^2}$$

$$= \underline{\underline{\frac{x^2+2x}{(x+1)^2}}}$$

N.R.
$u = x^2$
$v = x+1$
$u' = 2x$
$v' = 1$

$$\sim \frac{x(x+2)}{(x+1)^2}$$

$$\frac{x^2+2x}{x^2+2x+1}$$

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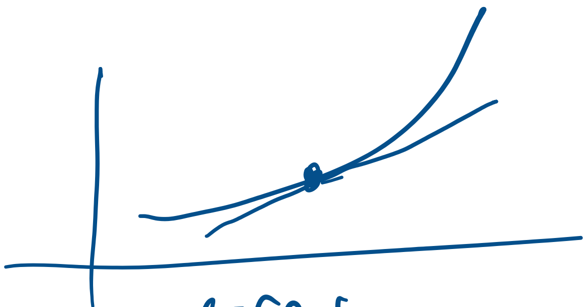
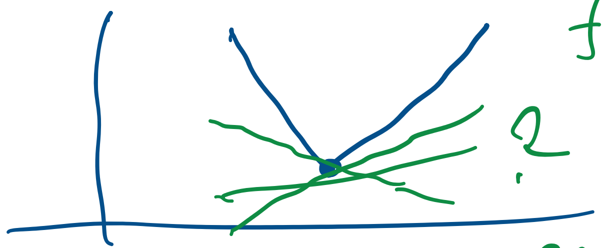
differenzierbar $\Leftrightarrow$ $\lim\limits_{h\to 0} \dfrac{f(x_0+h)-f(x_0)}{h}$ existiert



diff. bar

$f(x)=|x|$



nicht diff. bar an Knickstelle

Diff Ketten

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$$h(x) = \left(\frac{x^2}{x+1}\right)^2$$
$$= g(f(x))$$
$$h'(x) = g'(u)\Big|_{u=f(x)} \cdot f'(x)$$
$$= 2\,\frac{x^2}{x+1} \cdot \frac{x^2+2x}{(x+1)^2}$$
$$= \frac{2x^2(x^2+2x)}{(x+1)^3}$$

N.R.

$g(u) = u^2$

$f(x) = \frac{x^2}{x+1}$

$g'(u) = 2u$

$f'(x) = \frac{x^2+2x}{(x+1)^2}$ (s.o.)

Bsp 2:

$$h(x) = (\sin(x))^3$$
$$h'(x) = 3(\sin(x))^2 \cdot \cos(x)$$

N.R.

$g(u) = u^3$

$f(x) = \sin(x)$

$g'(u) = 3u^2$

$f'(x) = \cos(x)$

Diff Übung

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a) $f(x) = \dfrac{x^3}{(x+1)^2}$

Quotientenregel

N.R. $u = x^3$, $u' = 3x^2$

$v = (x+1)^2$

$v' = 2(x+1)$ (Kette)

$f'(x) = \dfrac{(x+1)^{\cancel{2}} \cdot 3x^2 - 2\cancel{(x+1)}x^3}{(x+1)^{\cancel{4}\,3}}$

$\dfrac{(x+1)3x^2 - 2x^3}{(x+1)^3}$

$= \dfrac{3x^3 + 3x^2 - 2x^3}{(x+1)^3}$

$= \underline{\underline{\dfrac{x^3 + 3x^2}{(x+1)^3}}}$

b) $h_1(x) = \sin(x^3)$

N.R. Kettenregel

$g(u) = \sin u$, $g'(u) = \cos u$

$f(x) = x^3$, $f'(x) = 3x^2$

$\underline{\underline{h_1'(x) = \cos(x^3) \cdot 3x^2}}$

c) $h(x) = \exp(\sin(x^2))$

$= e^{\sin(x^2)}$

N.R. Kettenregel mehrfach

$g(u) = e^u$, $g'(u) = e^u$

$f_1(v) = \sin(v)$, $f_1'(v) = \cos(v)$

$f_2(x) = x^2$, $f_2'(x) = 2x$

$u = \sin(v)$, $u' = \cos v$

$v = x^2$, $v' = 2x$

$h'(x) = \exp(\sin(x^2)) \cdot \cos(x^2) \cdot 2x$

$= \underline{\underline{e^{\sin(x^2)} \cos(x^2)\, 2x}}$

Taylor Bsp

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Bsp: $f(x) = \ln(x)$. Was ist $\ln(1.5)$?

Was ist Taylorpolynom der Ordnung $n = 3$ um $x_0 = 1$?

Lsg:

n	$f^{(n)}(x)$	$f^{(n)}(x_0) = f^{(n)}(1)$	
0	$\ln(x)$	0	$= a_0$
1	x^{-1}	1	$= a_1$
2	$-x^{-2}$	-1	$= a_2$
3	$2x^{-3}$	2	$= a_3$
$n+1 = 4$	$-6x^{-4}$		

$$P_3(x-1) = \underbrace{0}_{a_0} + \underbrace{1}_{a_1} \cdot (x-x_0)^1 + \frac{\overbrace{-1}^{a_2}}{2!}(x-x_0)^2 + \frac{2}{3!}(x-x_0)^3$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

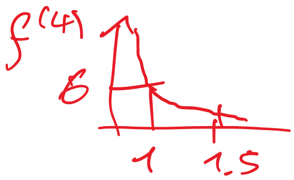
$$\ln(1.5) \approx P_3(1.5-1)$$

$$= 0.5 - \frac{1}{2}\, 0.5^2 + \frac{1}{3}\, 0.5^3$$

$$= \underline{\underline{0.41\overline{6}}}$$

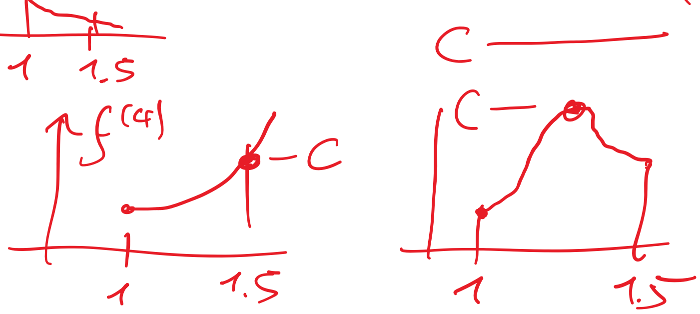
Restglied: $|f^{(n+1)}(x)| = |-6x^{-4}| = 6x^{-4} = \frac{6}{x^4}$

abschätzen im Bereich $[x_0, x_1] = [1, 1.5]$



Hier $C = 6$ $(= |f^{(4)}(1)|)$

Wäre



$$|R_3(1.5)| = \frac{6}{(3+1)!} \cdot |1.5-1|^{3+1}$$

$$= 0.0156$$

Probe: $|\ln(1.5) - 0.41\overline{6}| = 0.0112$

Taylor Ü

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$f(x) = \sin(x)$ um $x_0 = 0$ zum Grade 5 entwickeln

Lsg

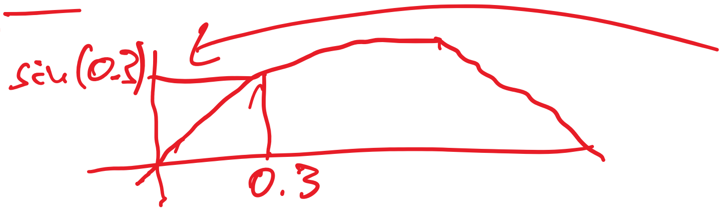
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin(x)$	0
1	$\cos(x)$	1
2	$-\sin(x)$	0
3	$-\cos(x)$	-1
4	$\sin(x)$	0
5	$\cos(x)$	1
6	$-\sin(x)$	

$$P_5(x) = \frac{1}{1!}x + \frac{-1}{3!}x^3 + \frac{1}{5!}x^5$$

$$P_5(0.3) = \underline{0.2955}$$

Restglied: $\underbrace{|f^{(6)}(x)|}_{\sin(x)}$ für $x \in [0, 0.3]$ Obere Schranke

$C = 1$ (weil $\sin(x) \leq 1 \quad \forall x$)

[

wäre genauere Schranke]

$$\text{Restglied } R = \frac{1}{6!}(0.3-0)^6 = \underline{1 \cdot 10^{-6}}$$

[Probe: $|0.2955 - \underbrace{\sin(0.3)}_{0.29552020}| = 2.02 \cdot 10^{-5}$]

$P_3(x)$

$\underbrace{\text{genauer: } 0.29552025}$

$\Rightarrow$ Fehler $4 \cdot 10^{-8}$