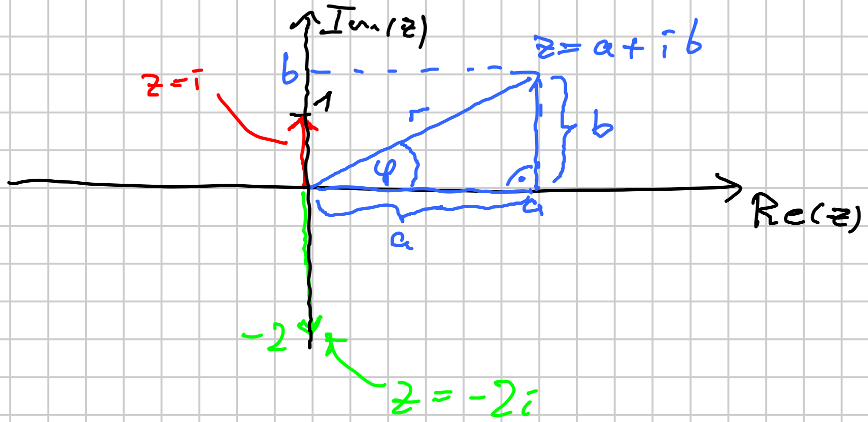
V MA 2 – 31.5.2017

Gaußsche Zahlenebene



$r = |z| = \sqrt{a^2 + b^2}$

Wie kann ich a und b durch r und φ ausdrücken

$$\sin\varphi = \frac{b}{r} \iff b = r\sin\varphi$$

$$\cos\varphi = \frac{a}{r} \iff a = r\cos\varphi$$

$$\begin{aligned} z &= a + ib \\ &= r(\cos\varphi + i\sin\varphi) \end{aligned}$$

trigonometr. Form

Begründung zu Euler $e^{i\varphi} = \cos\varphi + i\sin\varphi$

Taylorentwicklung von e^x mit $x = i\varphi$

$$e^{i\varphi} = 1 + \frac{(i\varphi)^1}{1!} + \frac{(i\varphi)^2}{2!} + \frac{(i\varphi)^3}{3!} + \frac{(i\varphi)^4}{4!} + \ldots$$

$$\underline{1} + \underline{i\frac{\varphi}{1!}} - \underline{\frac{\varphi^2}{2!}} - \underline{i\frac{\varphi^3}{3!}} + \underline{\frac{\varphi^4}{4!}} + \ldots$$

$$\cos\varphi = 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} + \ldots$$

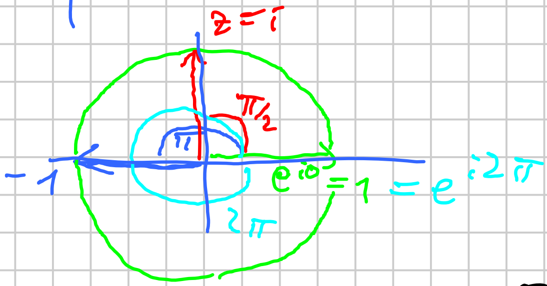
$$i\sin\varphi = i\frac{\varphi}{1!} - i\frac{\varphi^3}{3!} + \ldots$$

$$\Rightarrow e^{i\varphi} = \cos\varphi + i\sin\varphi$$

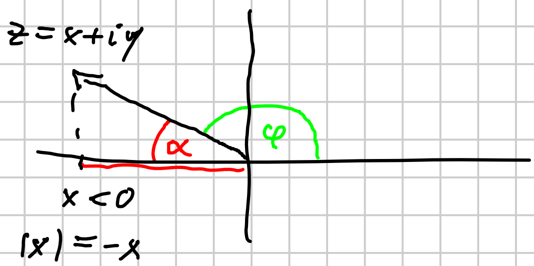
Übung

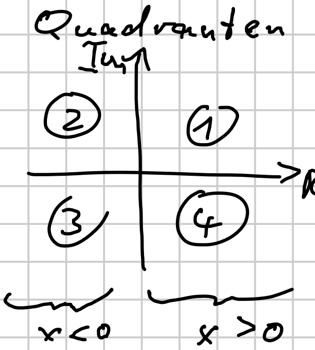
$$\mathrm{Re}(e^{i\varphi}) = \cos\varphi$$
$$\mathrm{Im}(e^{i\varphi}) = \sin\varphi$$

z	$\mathrm{Re}(z)$	$\mathrm{Im}(z)$	
e^{i0}	1	0	$e^{i0} = 1$
$e^{i\pi}$	-1	0	$e^{i\pi} = -1$
$e^{i2\pi}$	1	0	$e^{i2\pi} = 1$
$e^{i\pi/2}$	0	1	$e^{i\pi/2} = i$



Umrechnung "kartesisch → polar"





$$\tan\alpha = \frac{y}{|x|} = \frac{y}{-x} \Rightarrow \alpha = \arctan\left(\frac{y}{-x}\right) = -\arctan\left(\frac{y}{x}\right)$$

$$\varphi = \pi - \alpha = \pi + \arctan\left(\frac{y}{x}\right)$$

$$\varphi = \pi + \arctan\left(\frac{y}{x}\right)$$

$$\boxed{\arctan(x) \mathrel{\hat=} \mathrm{atan}(x) \mathrel{\hat=} \tan^{-1} \; (*)}$$

Übung $z = x + iy = \frac{3}{2} + i\left(-\frac{3}{2}\sqrt{3}\right)$ in Polarform

Lsg.: $x = \frac{3}{2} > 0 \Rightarrow$ ① oder ④

$y < 0 \Rightarrow$ ④ Quadrant

$$r = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\sqrt{3}\right)^2} = \sqrt{\left(\frac{3}{2}\right)^2\left(1 + \sqrt{3}^2\right)} =$$
$$= \frac{3}{2}\sqrt{1+3} = \underline{\underline{3}}$$

$$\varphi = \arctan\left(\frac{-\frac{3}{2}\sqrt{3}}{\left(\frac{3}{2}\right)}\right) = \arctan\left(-\sqrt{3}\right) = -\frac{\pi}{3}$$

$$-\frac{\pi}{3} \overset{?}{=} \frac{5\cdot 2\pi}{6} = \frac{5\pi}{3}$$

$$-\frac{\pi}{3} + 2\pi = -\frac{\pi}{3} + \frac{6\pi}{3} = \frac{5\pi}{3}$$

$$-\frac{\pi}{3} \;\hat{=}\; \frac{5\pi}{3} \quad \text{als Phase}$$

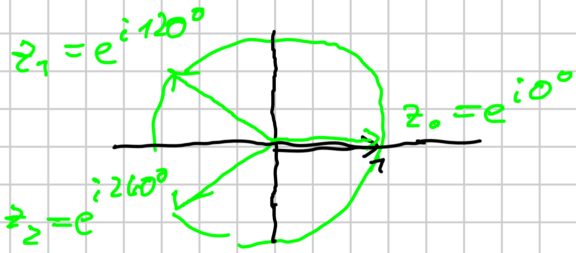
Potenzen komplexer Zahlen

Für welche $z = e^{i\varphi}$ gilt

$$z^3 = 1$$
$$(e^{i\varphi})^3 = 1$$
$$e^{i3\varphi} = 1$$

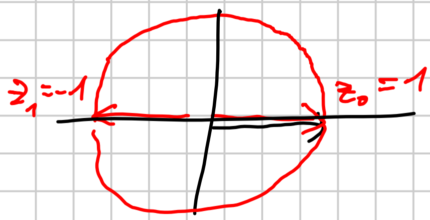
Welche φ erfüllen $3\varphi = 0° + k\cdot 360°$, $k \in \mathbb{Z}$

$$\varphi = 0° \Rightarrow 3\varphi = 0°$$
$$\varphi = 120° \Rightarrow 3\varphi = 360° \,\hat{=}\, 0° \; (k=1)$$
$$\varphi = 240° \Rightarrow 3\varphi = 720° \,\hat{=}\, 0° \; (k=2$$



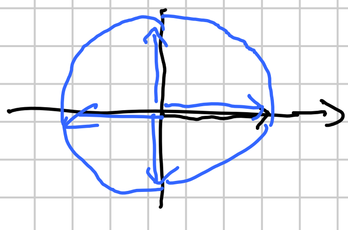
symmetrische 3-Teilung

Wie bei $z^2 = 1$?



symmetrische 2-Teilung

Wie bei $z^4 = 1$?



symmetrische 4-Teilung

Beispiel Potenzieren

$z = i$

Was ist $i^{\frac{5}{3}}$?

$p = 5$
$q = 3$

$$\boxed{i = e^{i\frac{\pi}{2}}}$$

$$z = i^{\frac{5}{3}} = \left(e^{i\frac{\pi}{2}}\right)^{\frac{5}{3}} = \left(e^{i\left(\frac{5\pi}{2} + 2k\pi\right)}\right)^{\frac{1}{3}}$$

$$= e^{i\left(\frac{5\pi}{6} + \frac{2k}{3}\pi\right)} \qquad k = 0, 1, 2$$

$$z_0 = e^{i\frac{5\pi}{6}} = \underline{\underline{-0.866 + 0.5\,i}}$$

$$z_1 = e^{i\left(\frac{5\pi}{6} + \frac{2\pi}{3}\right)} = e^{i\frac{9\pi}{6}} = \underline{\underline{-i}}$$

$$z_2 = e^{i\frac{13\pi}{6}} = \underline{\underline{0.866 + 0.5\,i}}$$