V MA2 — 22.05.2019

Komplexe Zahl Wdh

$$z = \underbrace{a}_{\text{Real-}} + i\underbrace{b}_{\text{Imaginär-}} \quad \text{-teil}$$

$$\boxed{i^2 = -1}$$

$z^* = a - ib$ konjugiert-komplexe Zahl zu z

$z \cdot z^* = a^2 + b^2$ rein reell

$r = |z| = \sqrt{a^2 + b^2}$ Betrag von z

Mehrdeutigkeit Polar

$$z = re^{i\varphi} = re^{i(\varphi + 2\pi)} = re^{i(\varphi + 4\pi)} = \dots$$

N.R.

$2\pi \mathrel{\hat{=}} 360°$

$-\frac{\pi}{2} = -90°$

Bsp: $\varphi = -\frac{\pi}{2}$, $r = 1$

$$z = e^{-i\pi/2} = e^{i(-\frac{\pi}{2} + 2\pi)} = e^{i3\pi/2}$$

$$= e^{-i\,90°} = e^{i\,270°}$$

Umrechnung polar → kartesisch

einfach: $x = r\cos\varphi$

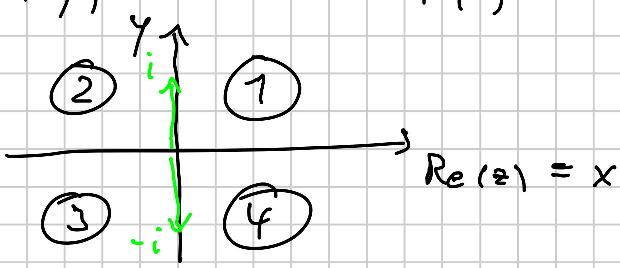
$y = r\sin\varphi$

Umrechnung kartesisch → polar

$(x, y) \rightarrow (r, \varphi)$

$\tan\varphi = \frac{y}{x}$

$r = \sqrt{x^2 + y^2}$



$\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ Wertebereich atan

$x < 0$ "$+\pi$"

$(x > 0)$ ohne "$+\pi$"

1) Bsp: $z = \underbrace{-2}_{x} + \underbrace{2\sqrt{3}}_{y}\, i$ $\qquad x<0,\ y>0$

$\Rightarrow$ 2. Quadrant

Gesucht: r und φ (Polarform)

$$r = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{2^2(1+3)} = 2\cdot 2 = 4$$

$$\varphi = \arctan\left(\frac{2\sqrt{3}}{-2}\right) + \pi = -\frac{\pi}{3} + \pi = \underline{\underline{\frac{2}{3}\pi}} \mathrel{\hat{=}} \underline{\underline{120^\circ}}$$

$$-60^\circ + 180^\circ = 120^\circ \qquad \underline{|\checkmark|}$$

2) Bsp: $z = 3e^{i\,5\cdot 2\pi/6}$

Gesucht: x und y (kartes. Form)

$$x = 3\cos\left(\frac{5}{6}2\pi\right) = 1.5$$

$$y = 3\sin\left(\frac{5}{6}2\pi\right) = -\frac{3}{2}\sqrt{3} = -2.598$$

Ü Aus x, y als Probe wieder r, φ errechnen

Gegeben $x = 1.5 = \frac{3}{2}$ $\qquad \left.\begin{matrix} x>0 \\ y<0 \end{matrix}\right\} \Rightarrow$ 4. Quadrant

$y = -\frac{3}{2}\sqrt{3}$

$$\Rightarrow \quad r = \sqrt{x^2+y^2} = \sqrt{\left(\frac{3}{2}\right)^2 + \left(-\frac{3}{2}\sqrt{3}\right)^2} = \frac{3}{2}\sqrt{1+3} = \underline{\underline{3}}$$

$$\varphi = \arctan\left(\frac{-\frac{3}{2}\sqrt{3}}{\frac{3}{2}}\right) = -\frac{\pi}{3} \mathrel{\hat{=}} \frac{5\pi}{3} = \underline{\underline{\frac{5}{6}2\pi}} \mathrel{\hat{=}} 300^\circ$$

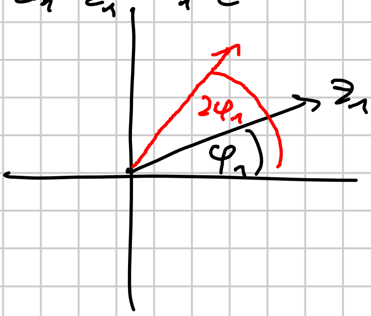
$$= -\frac{\pi}{3} + 2\pi = -\frac{\pi}{3} + \frac{6\pi}{3} = \frac{5\pi}{3}$$

Division in Exponentialform

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\varphi_1}}{r_2 e^{i\varphi_2}} = \left(\frac{r_1}{r_2}\right) e^{i\varphi_1} e^{-i\varphi_2} = \left(\frac{r_1}{r_2}\right) e^{i\varphi_1 - i\varphi_2}$$

$$= \left(\frac{r_1}{r_2}\right) e^{i(\varphi_1 - \varphi_2)}$$

$$z_1 \cdot z_1 = r_1^2 e^{i(\varphi_1+\varphi_1)} = r_1^2 e^{i2\varphi_1}$$



Für welche φ gilt bei $z = e^{i\varphi}$ dass $z^3 = 1$

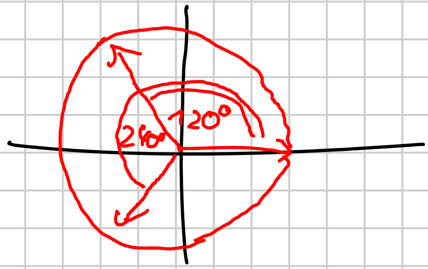
Lsg $\quad 3\varphi = k\,360°$

$\varphi = k\,120°$

also $\varphi_1 = 0°$

$\varphi_2 = 120°$

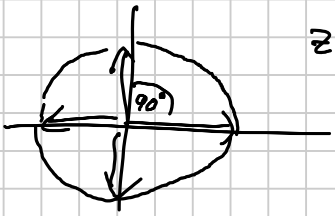
$\varphi_3 = 240°$



Für welche φ gilt

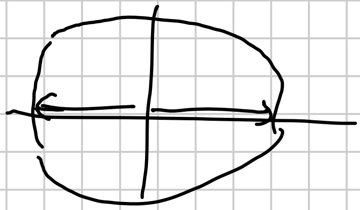
$z^4 = 1$

d.h. $z = 1, i, -1, -i$



$z^2 = 1$

d.h. $z = 1, -1$



Potenzieren mit c = rationale Zahl

Bsp: Was ist $z = i^{\frac{5}{3}}$?

Exponentialform: $\quad i = e^{i\pi/2} = e^{i90°}$

$$\frac{\pi}{2} \cdot \frac{5}{3} = \frac{5}{6}\pi$$

$$i^{\frac{5}{3}} = \left(e^{i\,90°}\right)^{5/3} = \left(e^{i(90° + k\,360°)}\right)^{5/3}$$

$$= e^{i(150° + k\cdot\cancel{5}\cdot 120°)} \qquad k = 0, 1, 2$$

$$120° = \frac{360°}{3}$$

Drei Lösungen

$$z_0 = e^{i\,150°} = -0.866 + 0.5i \qquad (k=0)$$

$$z_1 = e^{i\,270°} = -i \qquad (k=1)$$

$$z_2 = e^{i\,390°} = 0.866 + 0.5\,i \qquad (k=2)$$

Ü Berechne $\boxed{z = (1+i)^{\frac{3}{4}}}$

Wechsel Polar $\left.\begin{array}{l} r = ? \\ \varphi = ? \end{array}\right\}$ für $w = (1+i)$

$$\left.\begin{array}{l} r = \sqrt{1^2+1^2} = \sqrt{2} = 2^{1/2} \\ \varphi = \operatorname{atan}\left(\frac{1}{1}\right) = \frac{\pi}{4} \ \hat{=}\ 45° \end{array}\right\} \quad w = \sqrt{2}\, e^{i\pi/4}$$

$$z = \left(\sqrt{2}\, e^{i\pi/4}\right)^{\frac{3}{4}}$$

$$= \left(\sqrt{2}\, e^{i(\pi/4 + 2k\pi)}\right)^{\frac{3}{4}}$$

$$= \sqrt{2}^{\frac{3}{4}}\, e^{i(3\pi/16 + \cancel{3}k\,2\pi/4)}$$

$$= 2^{\frac{3}{8}}\, e^{i(3\pi/16 + k\cdot\frac{\pi}{2})}$$

$$k = 0,1,2,3$$