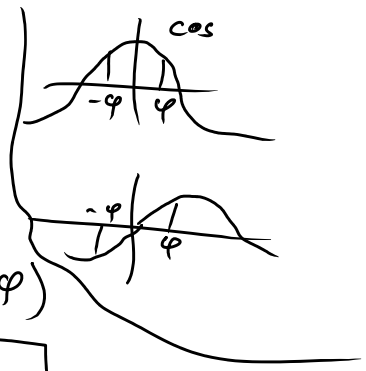
V2023-05-17

Mittwoch, 17. Mai 2023 10:56

Orga : letzte Klausureinsicht Mi 24.5., $12^{45}-13^{30}$, R3.230

Eulersche Formel

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$



Folgerung

$$e^{-i\varphi} = e^{i(-\varphi)} = \cos(-\varphi) + i\sin(-\varphi)$$

$$\boxed{e^{-i\varphi} = \cos\varphi - i\sin\varphi}$$

Beispiel $z = \underbrace{-2}_{x} + \underbrace{2\sqrt{3}}_{y} i$ von kartesisch in Polarform:

Realteil x ist neg. $\Rightarrow$ 2./3. Quadrant "$+\pi$"

Betrag: $r = \sqrt{x^2+y^2} = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4+12} = 4$

Phase: $\varphi = \arctan\left(\frac{2\sqrt{3}}{-2}\right) + \pi = -\frac{\pi}{3} + \pi = \frac{2}{3}\pi \triangleq 120°$

$$\Rightarrow z = 4\left(\cos\left(\tfrac{2}{3}\pi\right) + i\sin\left(\tfrac{2}{3}\pi\right)\right) \quad \text{(trig. Form)}$$

$$= 4e^{i\frac{2}{3}\pi} = 4e^{i120°} \quad \text{(Exp.-Form)}$$

Beispiel: $z = 3e^{i\,5\cdot\frac{2\pi}{6}}$ von Polarform in kart

$$x = Re(z) = 3\cos\left(5\tfrac{2\pi}{6}\right) = 1.5$$

$$y = Im(z) = 3\sin\left(5\tfrac{2\pi}{6}\right) = -\tfrac{3}{2}\sqrt{3} \approx -2.598$$

Übung Rechnen Sie von $z = x+iy$ wieder zurück in die Exp. form.

$$r = \sqrt{x^2+y^2} = \sqrt{(1.5)^2 + (-2.598)^2} \approx 3$$

Realteil $x > 0 \Rightarrow$ kein "$+\pi$"

$$\varphi = \arctan\left(\frac{y}{x}\right) \approx -1.05 \quad \checkmark$$

$$5\cdot\frac{2\pi}{6} = 300^\circ = -60^\circ$$
$$= -\frac{2\pi}{6} \approx -1.05$$

- Division

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\varphi_1}}{r_2 e^{i\varphi_2}} = \left(\frac{r_1}{r_2}\right) e^{i\varphi_1}\cdot e^{-i\varphi_2} = \left(\frac{r_1}{r_2}\right) e^{i(\varphi_1-\varphi_2)}$$

- "hoch n" z.B. $n=10$

$$(z_1)^{10} = \left(r_1 e^{i\varphi_1}\right)^{10} = r_1^{10}\left(e^{i\varphi_1}\right)^{10} = r_1^{10} e^{i\,10\varphi}$$

Potenz mit rationaler Zahl $c = \frac{p}{q}$, p, q teilerfremd

1) hoch p
2) "$+2k\pi$"
3) hoch $\frac{1}{q}$

$$z^{\frac{p}{q}} = \left(\left(re^{i\varphi} \right)^p \right)^{\frac{1}{q}} \quad 1)$$

$$= \left(r^p e^{i(\varphi p + 2k\pi)} \right)^{\frac{1}{q}} \quad 2)$$

$$= r^{\frac{p}{q}} e^{i\left(\frac{p\varphi}{q} + \frac{2k\pi}{q}\right)} \quad 3)$$

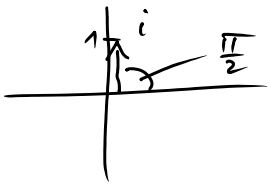
$$= r^c e^{i\left(c\varphi + \frac{2\pi k}{q}\right)}$$

$k = 0, 1, \ldots, q-1$

Bsp: Was ist $i^{\frac{5}{3}}$?

Lsg: $w = i \rightarrow$ Exp.darst.

$= 1 e^{i\frac{\pi}{2}}$

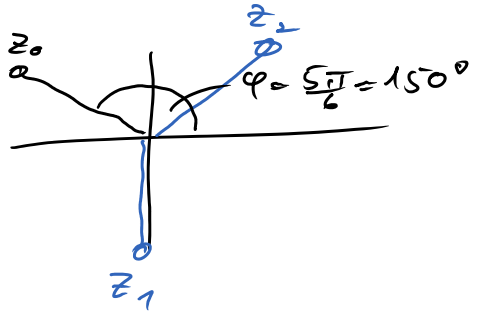


$$\boxed{i = e^{i\frac{\pi}{2}}}$$

$$i^{\frac{5}{3}} = \left(\left(e^{i\frac{\pi}{2}} \right)^5 \right)^{\frac{1}{3}} = \left(e^{i\left(\frac{5\pi}{2} + 2k\pi\right)} \right)^{1/3}$$

$$= e^{i\left(\frac{5\pi}{6} + \frac{2k\pi}{3}\right)}, \quad k = 0, 1, 2$$

Lsg in Exp.form



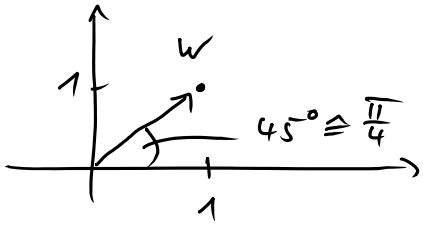
ii) $z = (1+i)^{\frac{3}{4}}$

(4) $z = (1+i)^{\frac{3}{4}}$ (c)

$z = w^c$

w

0) w in Exp. form $\quad w = 1+i$



$r = \sqrt{1^2+1^2} = \sqrt{2}$

$\varphi = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4} = 45°$

1) $z = \left(\left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)^3\right)^{1/4}$

2)

3)

$= \left(\sqrt{2}^3\, e^{i\left(\frac{\pi}{4}\cdot 3 + 2k\pi\right)}\right)^{1/4}$

$= 2^{\frac{3}{2\cdot 4}}\, e^{i\left(\frac{3\pi}{16} + \frac{2k\pi}{4}\right)}$ $\qquad k = 0, 1, 2, 3$

$= 2^{\frac{3}{8}}\, e^{i\left(\frac{3\pi}{16} + \frac{2k\pi}{4}\right)}$ $\qquad \frac{3\pi}{16} \approx 33°$